Charge-Conserving Hybrid Finite Element Methods for Maxwell's Equations and the Yang–Mills Equations

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Maxwell's equations in vacuum

Maxwell's equations in vacuum

Given charge density ρ and current density J satisfying $\dot{\rho} = -\operatorname{div} J$, solve

$$\dot{E} = \operatorname{curl} B - J, \qquad \dot{B} = -\operatorname{curl} E.$$

for the electric and magnetic fields E and B, subject to the constraints

$$\operatorname{div} E = \rho, \qquad \operatorname{div} B = 0.$$

Constraint preservation

If initial conditions satisfy constraints, then constraints satisfied for all time.

$$\frac{d}{dt}(\operatorname{div} E) = \operatorname{div} \dot{E} = \operatorname{div} \operatorname{curl} B - \operatorname{div} J = \dot{\rho},$$
$$\frac{d}{dt}(\operatorname{div} B) = \operatorname{div} \dot{B} = -\operatorname{div} \operatorname{curl} E = 0.$$

Maxwell's equations in a medium

Permittivity and permeability

- The electromagnetic properties of a medium are defined by scalar fields (or, more generally, matrix fields) ϵ and μ, the electric permittivity and magnetic permeability, respectively.
- We distinguish between the electric field E and the electric flux density

$$D := \epsilon E.$$

- We distinguish between the magnetic flux density B and the magnetic field

$$H := \mu^{-1} B.$$

Maxwell's equations

$$\dot{D} = \operatorname{curl} H - J, \qquad \dot{B} = -\operatorname{curl} E$$

$$\operatorname{div} D = \rho, \qquad \operatorname{div} B = 0.$$

Maxwell's equations in terms of potentials

Electric and magnetic potentials

- Let ϕ be a scalar field and A be a vector field, called the electric potential and magnetic potential respectively.
- Let
$$E := -(\dot{A} + \operatorname{grad}\phi), \qquad B := \operatorname{curl} A.$$
- E and B are invariant under the transformation
$$(\phi, A) \mapsto (\phi - \dot{\xi}, A + \operatorname{grad}\xi).$$
- Integrating $\dot{\xi} = \phi$ can WLOG set $\phi = 0$; this is the temporal gauge.

Maxwell's equations

$$\dot{D} = \operatorname{curl} H - J, \qquad \dot{B} = -\operatorname{curl} E$$
$$\operatorname{div} D = \rho, \qquad \operatorname{div} B = 0.$$

- Right equations automatically satisfied.
- Second-order equation in A: $\frac{d}{dt}(-\epsilon\dot{A}) = \operatorname{curl}(\mu^{-1}\operatorname{curl} A) - J$.

Nédélec's method

Maxwell's equations

$$\dot{D} = \operatorname{curl} H - J$$

Solve for A, where $D = -\epsilon \dot{A}$ and $H = \mu^{-1} \operatorname{curl} A$.

Weak formulation

$$\int_\Omega A' \cdot (\dot{D} + J) = \int_\Omega \operatorname{curl} A' \cdot H, \qquad \forall A' \in \mathring{H}(\operatorname{curl})$$

Solve for $A \in \mathring{H}(\operatorname{curl})$.

Galerkin semidiscretization

$$\int_\Omega A'_h \cdot (\dot{D}_h + J) = \int_\Omega \operatorname{curl} A'_h \cdot H_h, \qquad \forall A'_h \in V_h,$$

Solve for $A_h \in V_h$, where V_h a finite-dimensional subspace of $\mathring{H}(\operatorname{curl})$, $D_h = -\epsilon \dot{A}_h$, and $H_h = \mu^{-1} \operatorname{curl} A_h$.

- Second-order system of ODEs.

Nédélec's method: weak charge conservation

Nédélec's method

Solve

$$\int_\Omega A'_h \cdot (\dot{D}_h + J) = \int_\Omega \operatorname{curl} A'_h \cdot H_h, \qquad \forall A'_h \in V_h,$$

for $A_h \in V_h$, where V_h a finite-dimensional subspace of $\mathring{H}(\mathrm{curl})$, $D_h = -\epsilon \dot{A}_h$, and $H_h = \mu^{-1} \operatorname{curl} A_h$.

Weak charge conservation

- For all scalar fields ϕ'_h such that $\operatorname{grad} \phi'_h \in V_h$, set $A'_h = \operatorname{grad} \phi'_h$:
$$\int_\Omega \operatorname{grad} \phi'_h \cdot (\dot{D}_h + J) = 0.$$
- Weak form of charge conservation:
$$\operatorname{div} \dot{D} = -\operatorname{div} J = \dot{\rho}.$$
- If V_h is a space of curl-conforming Nédélec elements, then ϕ'_h piecewise polynomial up to degree r all satisfy $\operatorname{grad} \phi'_h \in V_h$.

Domain decomposition

Domain decomposition (see Brezzi and Fortin)

- Fix a triangulation $\mathcal{T}_h$; allow A to be discontinuous between simplices.
- Enforce continuity with Lagrange multipliers.

Weak formulation

$$\int_\Omega \Big(A' \cdot (\dot{D} + J) - \operatorname{curl} A' \cdot H \Big) = 0, \qquad \forall A' \in \mathring{H}(\operatorname{curl}; \Omega)$$

Solve for $A \in \mathring{H}(\operatorname{curl}; \Omega)$.

Domain-decomposed weak formulation

$$\int_K \Big(A' \cdot (\dot{D} + J) - \operatorname{curl} A' \cdot H \Big) + \int_{\partial K} (A' \times \widehat{H}) \cdot \mathbf{n} = 0, \quad \forall A' \in H(\operatorname{curl}; K)$$

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} (A \times \widehat{H}') \cdot \mathbf{n} = 0, \quad \forall \widehat{H}' \in H(\operatorname{curl}; \Omega).$$

Solve for $A \in H(\operatorname{curl}; K)$ $\forall K \in \mathcal{T}_h$ and $\widehat{H} \in H(\operatorname{curl}; \Omega)$.

Domain-decomposed Maxwell's equations

Domain-decomposed weak formulation

$$\int_K \Big(A' \cdot (\dot{D} + J) - \operatorname{curl} A' \cdot H \Big) + \int_{\partial K} (A' \times \widehat{H}) \cdot \mathbf{n} = 0, \quad \forall A' \in H(\operatorname{curl}; K)$$

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} (A \times \widehat{H}') \cdot \mathbf{n} = 0, \quad \forall \widehat{H}' \in H(\operatorname{curl}; \Omega).$$

Solve for $A \in H(\operatorname{curl}; K)\ \forall K \in \mathcal{T}_h$ and $\widehat{H} \in H(\operatorname{curl}; \Omega)$, where $D = -\epsilon \dot{A}$ and $H = \mu^{-1} \operatorname{curl} A$ (computed element-wise).

Proposition

A pair $(A, \widehat{H})$ solves the domain-decomposed problem if and only if A solves the original weak formulation and $\widehat{H} \times \mathbf{n}|_{\partial K} = H \times \mathbf{n}|_{\partial K}$ for all K.

$\widehat{D}$

If we do not gauge fix $\phi = 0$, then we also get Lagrange multiplier $\widehat{D}$ enforcing continuity of ϕ, and $\widehat{D} \cdot \mathbf{n}|_{\partial K} = D \cdot \mathbf{n}|_{\partial K}$ for all K.

Charge conservation

Semidiscretized domain-decomposed Maxwell's equations

$$\int_K \left(A'_h \cdot (\dot{D}_h + J) - \operatorname{curl} A'_h \cdot H_h \right) + \int_{\partial K} (A'_h \times \widehat{H}_h) \cdot \mathbf{n} = 0, \quad \forall A'_h \in V_h(K)$$

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} (A_h \times \widehat{H}'_h) \cdot \mathbf{n} = 0, \quad \forall \widehat{H}'_h \in \widehat{V}_h(\Omega).$$

Solve for $A_h \in V_h(K)\ \forall K \in \mathcal{T}_h$ and $\widehat{H}_h \in \widehat{V}_h(\Omega)$, where $V_h(K)$ and $\widehat{V}_h(\Omega)$ are finite-dimensional subspaces of $H(\operatorname{curl}; K)$ and $H(\operatorname{curl}; \Omega)$, respectively, $D_h = -\epsilon \dot{A}_h$, and $H_h = \mu^{-1} \operatorname{curl} A_h$ (computed element-wise).

- For large $\widehat{V}_h(\Omega)$, equivalent to Nédélec's method plus postprocessing.

$\widehat{D}_h$

- $\widehat{D}_h$ is the Lagrange multiplier enforcing continuity of ϕ_h.
- $\widehat{H}_h$ is in $H(\operatorname{curl})$, (unlike H_h), so let $\dot{\widehat{D}}_h = \operatorname{curl} \widehat{H}_h - J$.
- $\operatorname{div} \dot{\widehat{D}}_h = -\operatorname{div} J = \dot{\rho}$.

Numerical experiments

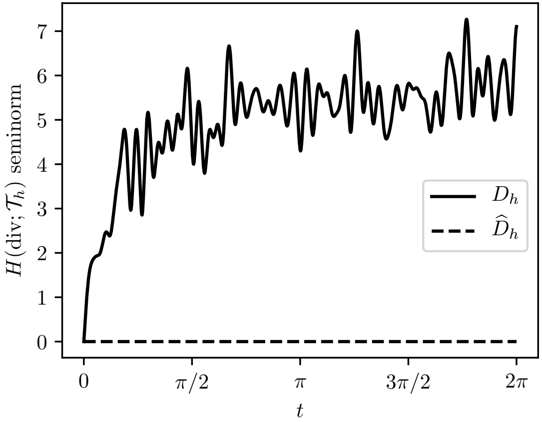

Figure: On a cube domain starting with no charge: total absolute charge using $D_h := -\epsilon \dot{A}_h$ (Nédélec's method, solid line) vs. $\widehat{D}_h$ (our method, dashed line).

Maxwell's equations using differential forms

Potentials, fields, fluxes

$$\phi \in \Lambda^0(\Omega), \qquad A \in \Lambda^1(\Omega),$$
$$E = -\dot{A} + d\phi \in \Lambda^1(\Omega) \qquad B = dA \in \Lambda^2(\Omega),$$
$$\epsilon \colon \Lambda^1 \to \Lambda^2 \qquad \mu \colon \Lambda^1 \to \Lambda^2,$$
$$D = \epsilon E \in \Lambda^2(\Omega), \qquad H = \mu^{-1} B \in \Lambda^1(\Omega),$$
$$\rho \in \Lambda^3(\Omega), \qquad J \in \Lambda^2(\Omega).$$

Maxwell's equations

$$\dot{D} = dH - J, \qquad \int_\Omega A' \wedge (\dot{D} + J) = \int_\Omega dA' \wedge H, \quad \forall A'$$
$$dD = \rho, \qquad \int_\Omega d\phi' \wedge D = \int_\Omega \phi' \rho, \quad \forall \phi'.$$

The Yang–Mills equations

Potentials, fields, fluxes

$$\phi \in \Lambda^0(\Omega, \mathfrak{g}), \qquad A \in \Lambda^1(\Omega, \mathfrak{g}),$$

$$E = -\dot{A} + d\phi + [A, \phi] \in \Lambda^1(\Omega, \mathfrak{g}) \qquad B = dA + \tfrac{1}{2}[A \wedge A] \in \Lambda^2(\Omega, \mathfrak{g}),$$

$$\epsilon \colon \Lambda^1 \to \Lambda^2 \qquad \mu \colon \Lambda^1 \to \Lambda^2,$$

$$D = \epsilon E \in \Lambda^2(\Omega, \mathfrak{g}), \qquad H = \mu^{-1} B \in \Lambda^1(\Omega, \mathfrak{g}),$$

$$\rho \in \Lambda^3(\Omega, \mathfrak{g}), \qquad J \in \Lambda^2(\Omega, \mathfrak{g}).$$

The Yang–Mills equations with $\phi = 0$ and $J = 0$

$$\dot{D} = dH + [A \wedge H], \qquad \int_\Omega \langle A' \wedge \dot{D} \rangle = \int_\Omega \langle (dA' + [A \wedge A']) \wedge H \rangle,$$

$$dD + [A \wedge D] = \rho, \qquad \int_\Omega \langle (d\phi' + [A, \phi'] \wedge D \rangle = \int_\Omega \langle \phi', \rho \rangle.$$

Nédélec's method only conserves total charge on Ω

see Christiansen and Winther

Semidiscretization of the Yang–Mills equations

$$\int_\Omega \langle A'_h \wedge \dot{D}_h \rangle = \int_\Omega \langle (dA'_h + [A_h \wedge A'_h]) \wedge H_h \rangle, \quad \forall A'_h \in V_h,$$

where V_h is a finite-dimensional subspace of $\mathring{H}\Lambda^1(\Omega, \mathfrak{g})$ (e.g. $\mathfrak{g}$-valued Nédélec elements).

Weak charge conservation

- For $\phi'_h \in \Lambda^0(\Omega, \mathfrak{g})$ such that $A'_h := d\phi'_h + [A_h, \phi'_h] \in V_h$, we have weak charge conservation

$$\int_\Omega \langle (d\phi'_h + [A_h, \phi'_h]) \wedge \dot{D}_h \rangle = 0 = \int_\Omega \langle \phi'_h, \dot{\rho} \rangle.$$

  - Weak form of $\frac{d}{dt}(dD_h + [A_h \wedge D_h])$
- Problem: $d\phi'_h + [A_h, \phi'_h]$ is generally only going to be in V_h if ϕ'_h is constant $\Rightarrow$ conservation only of total charge on Ω.

Our method conserves total charge on each element K

Semidiscretized domain-decomposed Yang–Mills equations

$$\int_K \Big(\langle A'_h \wedge \dot{D}_h \rangle - \langle (dA'_h + [A_h \wedge A'_h]) \wedge H_h \rangle \Big) + \int_{\partial K} \langle A'_h \wedge \widehat{H}_h \rangle = 0, \ \forall A'_h \in V_h(K),$$

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} \langle A_h \wedge \widehat{H}'_h \rangle = 0, \ \forall \widehat{H}'_h \in \widehat{V}_h(\Omega).$$

Solve for $A_h \in V_h(K)\ \forall K \in \mathcal{T}_h$ and $\widehat{H}_h \in \widehat{V}_h(\Omega)$, where $V_h(K)$ and $\widehat{V}_h(\Omega)$ are finite-dimensional subspaces of $H\Lambda^1(K, \mathfrak{g})$ and $H\Lambda^1(\Omega, \mathfrak{g})$, respectively, $D_h = -\epsilon \dot{A}_h$, and $H_h = \mu^{-1}(dA_h + \frac{1}{2}[A_h \wedge A_h])$ (computed element-wise).

Local charge conservation

- Let $\widehat{D}_h$ satisfy $\dot{\widehat{D}}_h = d\widehat{H}_h + [A_h \wedge \widehat{H}_h]$.
- No strong charge conservation: $\frac{d}{dt}(d\widehat{D}_h + [A_h \wedge \widehat{D}_h]) \neq 0$ (due to nonlinearity and $H_h \neq \widehat{H}_h$).
- Do have $\frac{d}{dt} \int_K \langle \phi'_h, d\widehat{D}_h + [A_h \wedge D_h] \rangle = 0$ for all ϕ'_h constant on K.
  - $\Rightarrow$ conservation of total charge on each element.

Numerical experiments

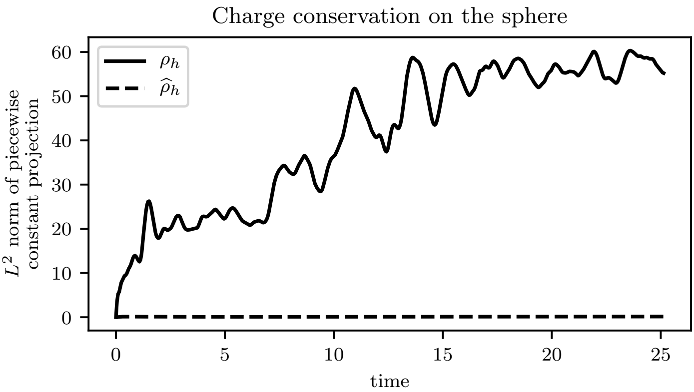

Figure: A simulation of the Yang–Mills equations with $\rho = 0$ and $\mathfrak{g} = \mathfrak{su}(2)$.

- Two estimates for charge:
  - $\rho_h := d D_h + [A_h \wedge D_h]$.
  - $\widehat{\rho}_h := d \widehat{D}_h + [A_h \wedge D_h]$.
- Plot: Average ρ_h ($\widehat{\rho}_h$) on each element K, then square and integrate.

Thank you

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