The Combinatorics of Finite Element Methods

Yakov Berchenko-Kogan

Florida Institute of Technology

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Outline

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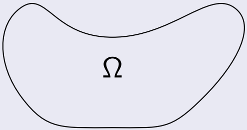
Sample Problem



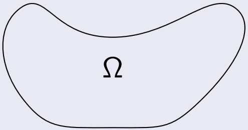
Numerically solving PDEs

Sample Problem

- Given $f\colon \Omega \to \mathbb{R}$, find $u\colon \Omega \to \mathbb{R}$ such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f$$

and u vanishes on the boundary.



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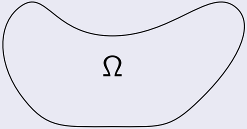

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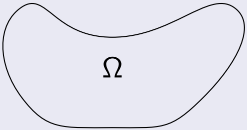

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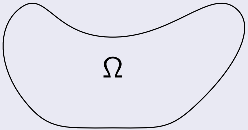
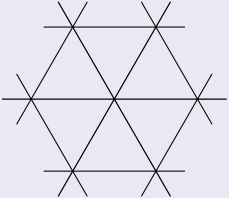


Discretization

- To solve numerically, we must discretize.
- We need a finite-dimensional space of functions that "approximates" the full infinite-dimensional space of possible u.

Finite-dimensional function spaces

Continuous piecewise linear functions to $\mathbb{R}$



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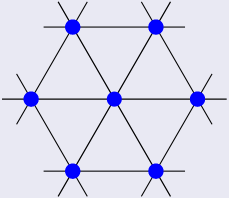

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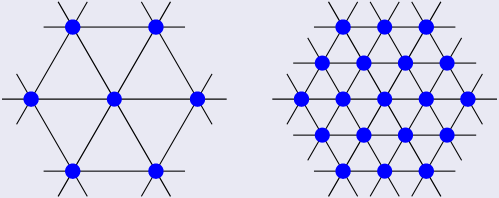

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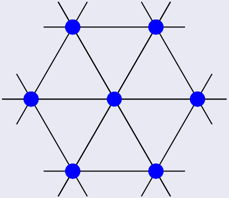

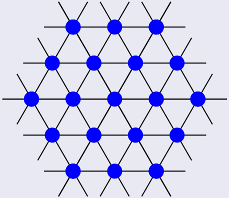

Continuous piecewise polynomial functions to $\mathbb{R}$

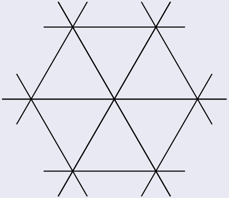

Figure: Piecewise quadratic (left) and piecewise cubic (right)

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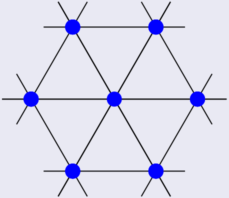

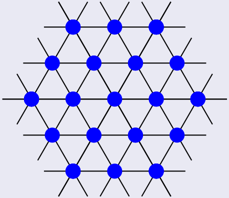

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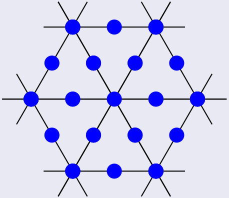

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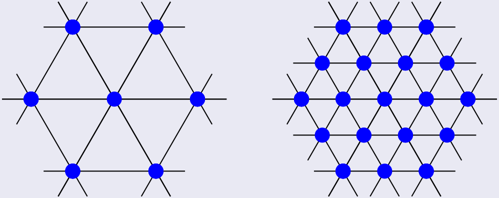

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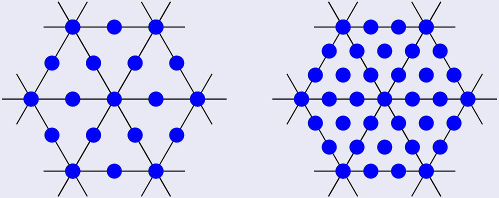

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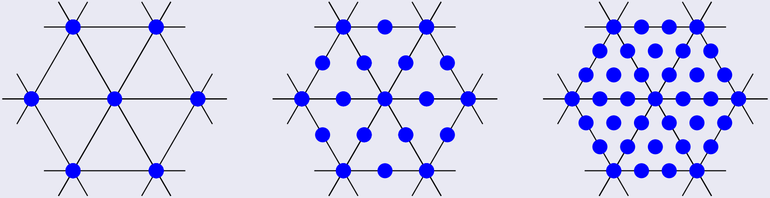
Degrees of freedom

Piecewise linear/quadratic/cubic continuous scalar-valued functions



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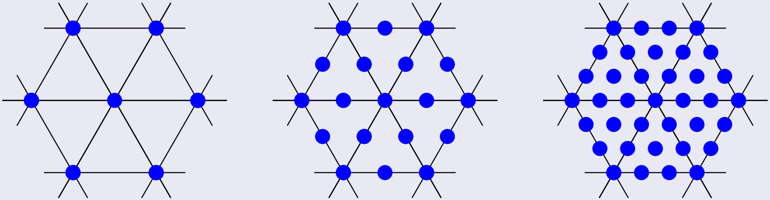

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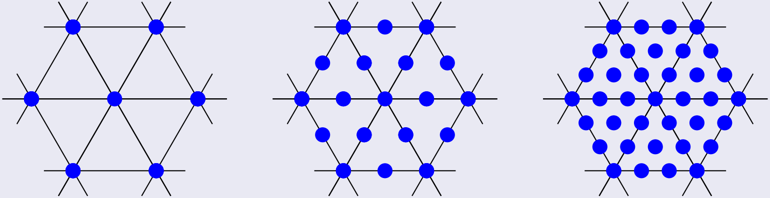

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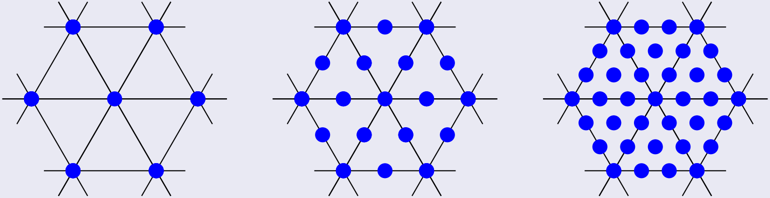

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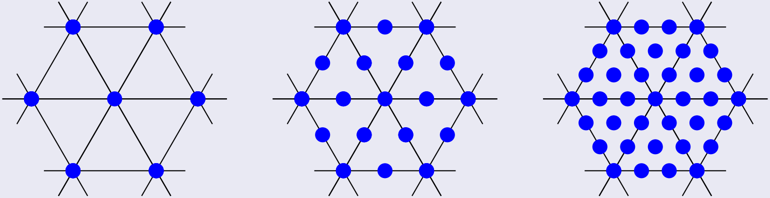

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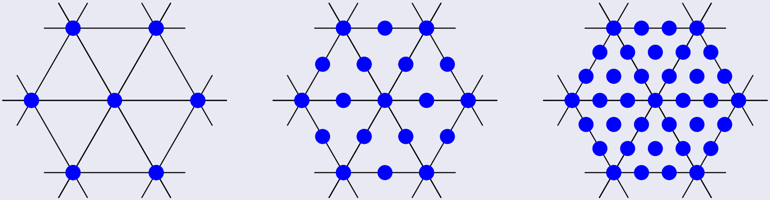

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Piecewise linear	$\mathbb{R}^{V}$
Piecewise quadratic	$\mathbb{R}^{V+E}$
Piecewise cubic	$\mathbb{R}^{V+2E+F}$

Finite-dimensional spaces of vector fields

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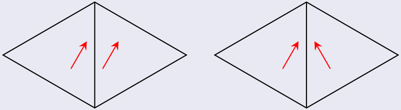

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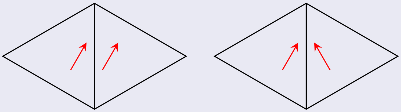

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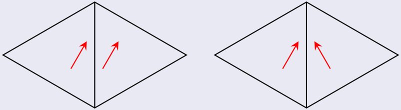

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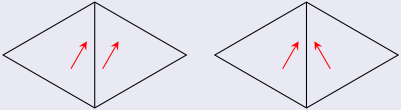

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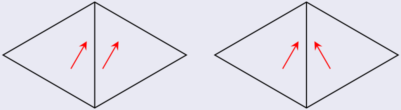

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  - Having well-defined line integrals requires only tangential continuity.

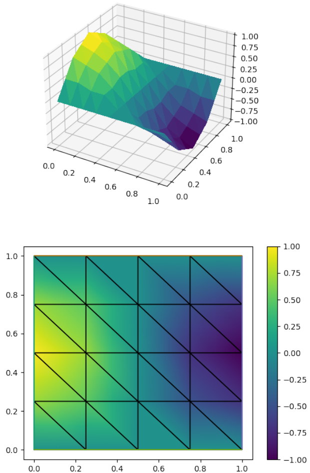
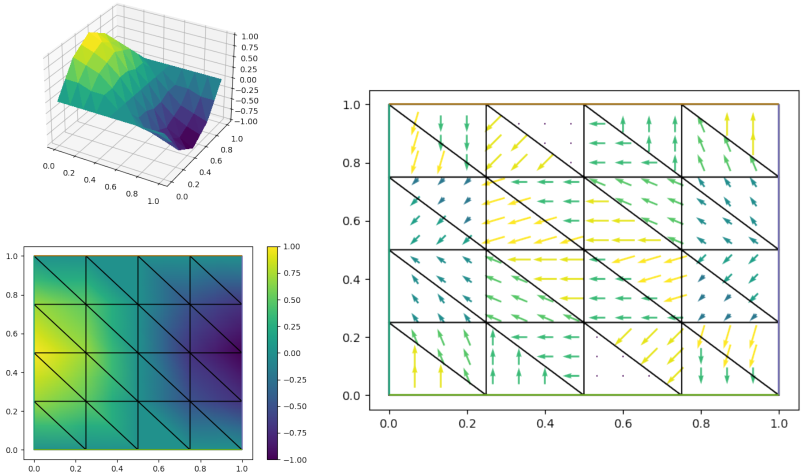


Figure: A piecewise linear function (left) and its gradient (right)

Gradients of piecewise smooth scalar fields



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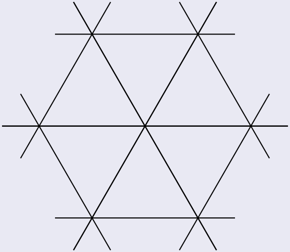

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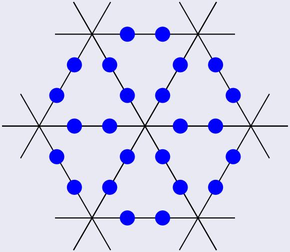

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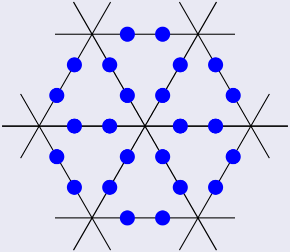

Higher degree?

Periodic Table of the Finite Elements

Complexes and cohomology

A discrete subcomplex of the de Rham complex

$$\begin{matrix}\text{continuous}\\ \text{piecewise cubic}\\ \text{scalar fields}\end{matrix} \xrightarrow{\text{grad}} \begin{matrix}\text{tangentially continuous}\\ \text{piecewise quadratic}\\ \text{vector fields}\end{matrix} \xrightarrow{\text{curl}} \begin{matrix}\text{discontinuous}\\ \text{piecewise linear}\\ \text{scalar fields}\end{matrix}$$

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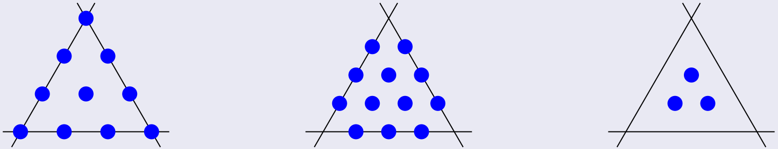
continuous piecewise cubic scalar fields $\xrightarrow{\text{grad}}$ tangentially continuous piecewise quadratic vector fields $\xrightarrow{\text{curl}}$ discontinuous piecewise linear scalar fields



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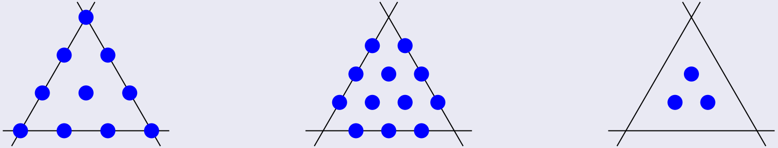

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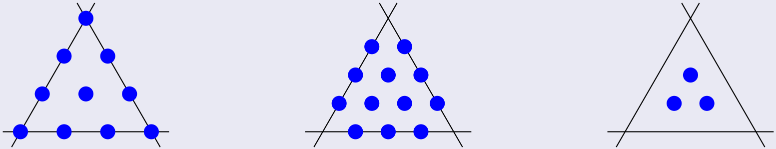

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Euler characteristic and cohomology of triangulated surfaces

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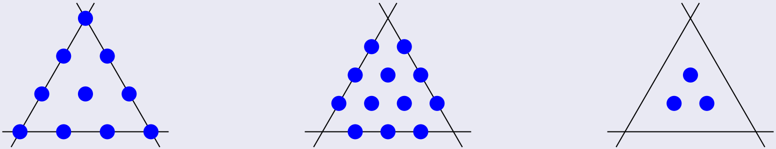

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Euler characteristic and cohomology of triangulated surfaces

- This complex has the right Euler characteristic:

$$(V + 2E + F) - (3E + 3F) + 3F = V - E + F.$$

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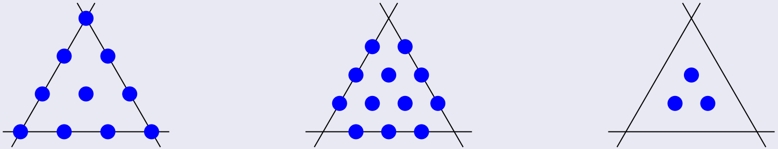

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Euler characteristic and cohomology of triangulated surfaces

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$$(V + 2E + F) - (3E + 3F) + 3F = V - E + F.$$

- The cohomology agrees with simplicial/de Rham cohomology.
  - (Arnold, Falk, Winther, 2010).

Whitney forms (Whitney, 1957)

Can we get simplicial cochains?

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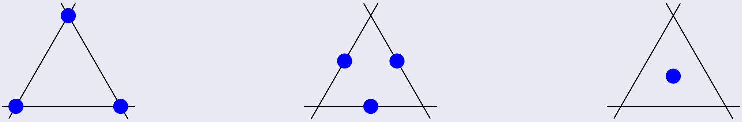

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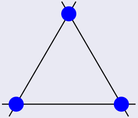
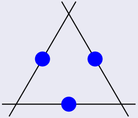
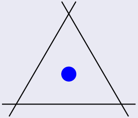

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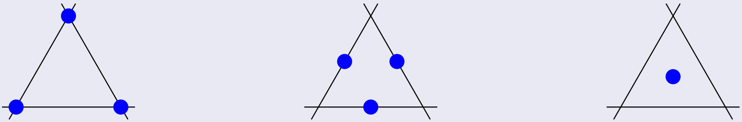

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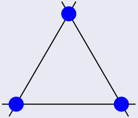
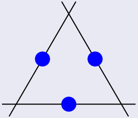
Barycentric coordinates (the standard simplex)

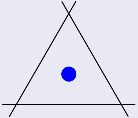
$$\{(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3_{\geq 0} \\ \mid \lambda_1 + \lambda_2 + \lambda_3 = 1\}$$

Can we get simplicial cochains?

$$\mathbb{R}^V \longrightarrow \mathbb{R}^E \longrightarrow \mathbb{R}^F$$







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Whitney one-forms:

$$\lambda_1 \, d\lambda_2 - \lambda_2 \, d\lambda_1,$$
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$\mathcal{P}_r\Lambda^n(\mathcal{T})$	discontinuous piecewise polynomial scalar fields

Complexes revisited

We've seen

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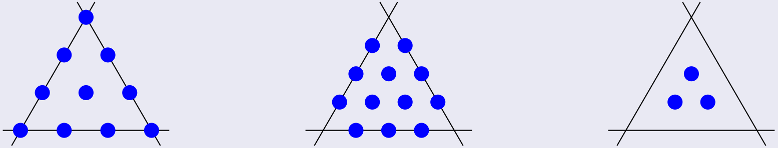
continuous piecewise cubic scalar fields $\xrightarrow{\text{grad}}$ tangentially continuous piecewise quadratic vector fields $\xrightarrow{\text{curl}}$ discontinuous piecewise linear scalar fields



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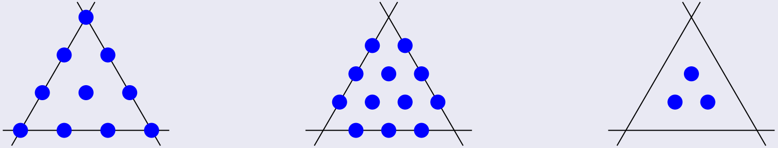

$$\mathcal{P}_3\Lambda^0(\mathcal{T}) \xrightarrow{\ d\ } \mathcal{P}_2\Lambda^1(\mathcal{T}) \xrightarrow{\ d\ } \mathcal{P}_1\Lambda^2(\mathcal{T})$$

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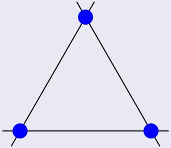

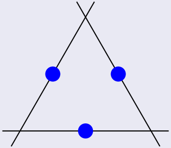

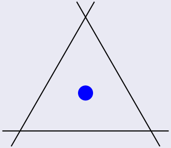

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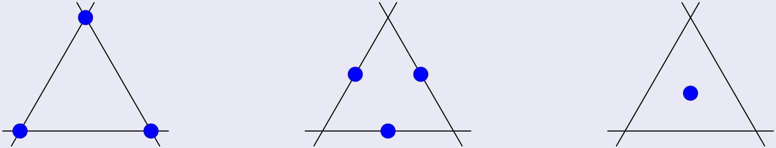

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For a triangulation $\mathcal{T}$, the cohomology of the complexes

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Theorem (Arnold, Falk, Winther, 2006)

We can "mix and match" using any of the maps

$$\mathcal{P}_r\Lambda^k(\mathcal{T}) \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^{k+1}(\mathcal{T}), \qquad \mathcal{P}_r\Lambda^k(\mathcal{T}) \xrightarrow{d} \mathcal{P}_r^-\Lambda^{k+1}(\mathcal{T})$$

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References

Douglas N. Arnold, Richard S. Falk, and Ragnar Winther.
Finite element exterior calculus, homological techniques, and applications.
Acta Numer., 15:1–155, 2006.

Douglas N. Arnold, Richard S. Falk, and Ragnar Winther.
Finite element exterior calculus: from Hodge theory to numerical stability.
Bull. Amer. Math. Soc. (N.S.), 47(2):281–354, 2010.

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- Get a finite-dimensional linear system of equations.

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Eigenvalues of the curl curl operator

On a square domain, find a vector field u (with appropriate boundary conditions) such that $\operatorname{curl}\operatorname{curl} u = \lambda u$.

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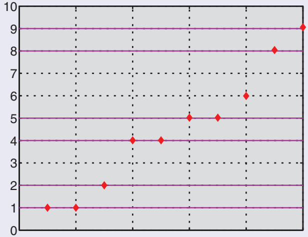

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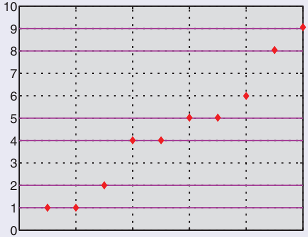

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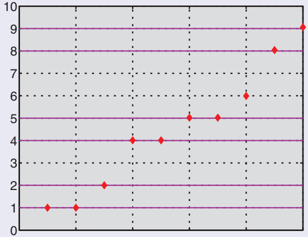

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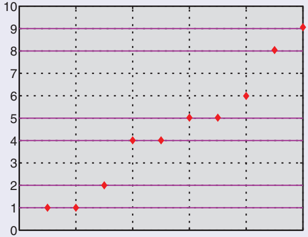

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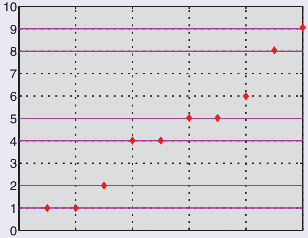

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- Can control if $\ker \operatorname{curl} = \operatorname{im} \operatorname{grad}$ holds on the discrete level.

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  - Otherwise, the quantities will be conserved only approximately and may drift over time.

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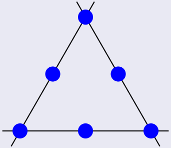
- can conserve ρ even in discrete setting (—, Stern, 2021).

Bases for scalar fields



Bases for scalar fields

- Recall barycentric coordinates:

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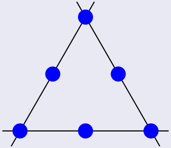

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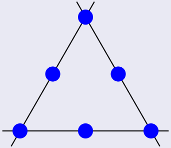

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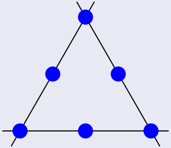

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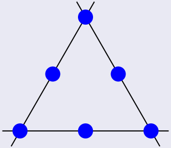

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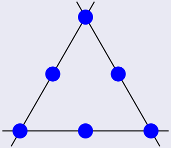

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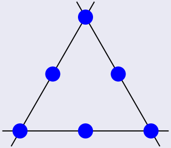

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  - In 2D and 3D, depends on the type of finite element space (e.g. $\mathcal{P}\Lambda^1$, $\mathcal{P}^-\Lambda^2$), and the polynomial degree modulo 3 (Licht, 2019; —, 2023).

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Further directions

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    - various combinations of —, Gawlik, Neunteufel, and others; 2019–2023 and in preparation.

Thank you