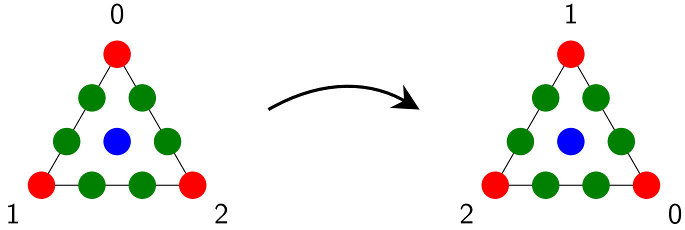
Duality and Symmetry in Finite Element Exterior Calculus

Yakov Berchenko-Kogan

Pennsylvania State University

June 19–25, 2022

Symmetry of Scalar Elements



$$\mathcal{P}_3\Lambda^0(T^2) = \left\langle \lambda_0^3, \lambda_1^3, \lambda_2^3, \lambda_1^2\lambda_2, \lambda_2^2\lambda_1, \lambda_2^2\lambda_0, \lambda_0^2\lambda_2, \lambda_0^2\lambda_1, \lambda_1^2\lambda_0, \lambda_0\lambda_1\lambda_2 \right\rangle .$$

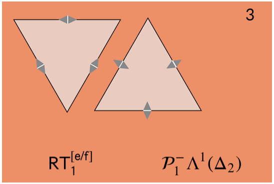
- When computing matrix of, e.g., $a(u,v) = \int_{T^2} \nabla u \cdot \nabla v$, can exploit sixfold symmetry of T^2 to compute fewer entries.

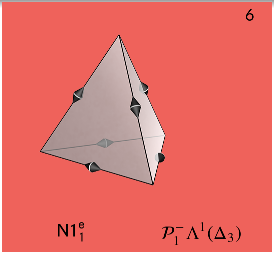
$$\begin{aligned} a\left(\lambda_0^3, \lambda_1^2\lambda_2\right) &= a\left(\lambda_1^3, \lambda_2^2\lambda_0\right) = a\left(\lambda_2^3, \lambda_0^2\lambda_1\right) \\ &= a\left(\lambda_0^3, \lambda_2^2\lambda_1\right) = a\left(\lambda_1^3, \lambda_0^2\lambda_2\right) = a\left(\lambda_2^3, \lambda_1^2\lambda_0\right) \end{aligned}$$

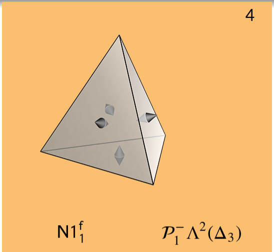
- More generally,
  $\int_{T^2} g^{-1}(du \otimes dv)\sqrt{\det g} = \sqrt{\det g}\, g^{-1}\left(\int_{T^2} du \otimes dv\right)$.

Symmetry of Vector Elements

Whitney Elements







$$\Big\langle \lambda_1\,d\lambda_2 - \lambda_2\,d\lambda_1,\quad \lambda_2\,d\lambda_0 - \lambda_0\,d\lambda_2,\quad \lambda_0\,d\lambda_1 - \lambda_1\,d\lambda_0 \Big\rangle.$$

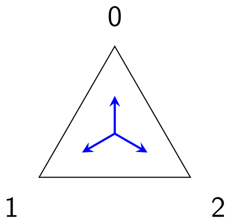
$$\Big\langle \lambda_1\,d\lambda_2 - \lambda_2\,d\lambda_1,\quad \lambda_2\,d\lambda_0 - \lambda_0\,d\lambda_2,\quad \lambda_0\,d\lambda_1 - \lambda_1\,d\lambda_0,\quad \lambda_0\,d\lambda_3 - \lambda_3\,d\lambda_0,\quad \lambda_1\,d\lambda_3 - \lambda_3\,d\lambda_1,\quad \lambda_2\,d\lambda_3 - \lambda_3\,d\lambda_2 \Big\rangle.$$

$$\Big\langle \lambda_1\,d\lambda_2\wedge d\lambda_3 + \lambda_2\,d\lambda_3\wedge d\lambda_1 + \lambda_3\,d\lambda_1\wedge d\lambda_2,\quad \dots,\quad \lambda_0\,d\lambda_1\wedge d\lambda_2 + \lambda_1\,d\lambda_2\wedge d\lambda_0 + \lambda_2\,d\lambda_0\wedge d\lambda_1 \Big\rangle$$

Geometric symmetry $\Rightarrow$ basis symmetry (up to sign).

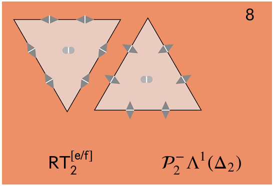
Symmetry of Vector Elements

Lack of Symmetric Bases

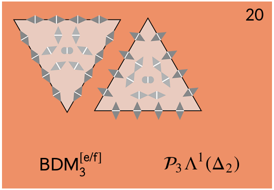


$$\mathcal{P}_0\Lambda^1(T^2) = \langle d\lambda_0, d\lambda_1, d\lambda_2 \rangle,$$

$$d\lambda_0 + d\lambda_1 + d\lambda_2 = 0$$



$$\langle \lambda_1^2\, d\lambda_2 - \lambda_1\lambda_2\, d\lambda_1,$$
$$\lambda_2^2\, d\lambda_1 - \lambda_1\lambda_2\, d\lambda_2,$$
$$\dots,$$
$$\lambda_0^2\, d\lambda_1 - \lambda_0\lambda_1\, d\lambda_0,$$
$$\lambda_1^2\, d\lambda_0 - \lambda_0\lambda_1\, d\lambda_1,$$
$$\lambda_0\lambda_1\, d\lambda_2 - \lambda_0\lambda_2\, d\lambda_1,$$
$$\lambda_1\lambda_2\, d\lambda_0 - \lambda_0\lambda_1\, d\lambda_2,$$
$$\lambda_0\lambda_2\, d\lambda_1 - \lambda_1\lambda_2\, d\lambda_0 \rangle.$$



$$\langle \quad \dots,$$
$$\dots,$$
$$\lambda_0\lambda_1\lambda_2\, d\lambda_0,$$
$$\lambda_0\lambda_1\lambda_2\, d\lambda_1,$$
$$\lambda_0\lambda_1\lambda_2\, d\lambda_2 \rangle.$$

Results

Theorem (if: Licht, 2019; only if: YBK, 2021)

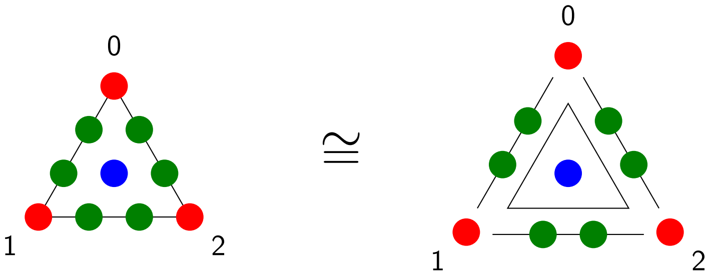
The following spaces have symmetry-invariant bases up to sign if and only if the corresponding condition holds.

$$\begin{aligned} &\mathcal{P}_r\Lambda^1(T^2) && \textit{if and only if} && r \notin 3\mathbb{N}_0, \\ &\mathcal{P}_r^-\Lambda^1(T^2) && \textit{if and only if} && r \notin 3\mathbb{N}_0 + 2. \end{aligned}$$

Theorem (YBK, 2021)

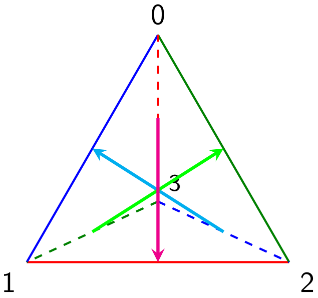
The following spaces have symmetry-invariant bases up to sign if and only if the corresponding condition holds.

$$\begin{aligned} &\mathcal{P}_r\Lambda^1(T^3) && \textit{always}, && \\ &\mathcal{P}_r^-\Lambda^1(T^3) && \textit{if and only if} && r \notin 3\mathbb{N}_0 + 2, \\ &\mathcal{P}_r\Lambda^2(T^3) && \textit{always}, && \\ &\mathcal{P}_r^-\Lambda^2(T^3) && \textit{always}. && \end{aligned}$$



$$\mathcal{P}_3\Lambda^0(T^2) \cong 3\mathring{\mathcal{P}}_3\Lambda^0(T^0) \oplus 3\mathring{\mathcal{P}}_3\Lambda^0(T^1) \oplus \mathring{\mathcal{P}}_3\Lambda^0(T^2)$$
$$\cong 3\mathcal{P}_2\Lambda^0(T^0) \oplus 3\mathcal{P}_1\Lambda^1(T^1) \oplus \mathcal{P}_0\Lambda^2(T^2)$$

$$\langle\lambda_0^3\rangle \oplus \langle\lambda_1^3\rangle \oplus \langle\lambda_2^3\rangle$$
$$\oplus \langle\lambda_1^2\lambda_2, \lambda_2^2\lambda_1\rangle \oplus \langle\lambda_2^2\lambda_0, \lambda_0^2\lambda_2\rangle \oplus \langle\lambda_0^2\lambda_1, \lambda_1^2\lambda_0\rangle \oplus \langle\lambda_0\lambda_1\lambda_2\rangle$$
$$\cong \langle\lambda_0^2\rangle \oplus \langle\lambda_1^2\rangle \oplus \langle\lambda_2^2\rangle$$
$$\oplus \langle\lambda_1\, ds, \lambda_2\, ds\rangle \oplus \langle\lambda_0\, ds, \lambda_2\, ds\rangle \oplus \langle\lambda_0\, ds, \lambda_1\, ds\rangle \oplus \langle 1\, dA\rangle$$



$$\mathcal{P}_0\Lambda^1(T^3)$$
$$= \langle d\lambda_0 + d\lambda_1 - d\lambda_2 - d\lambda_3,$$
$$d\lambda_0 + d\lambda_2 - d\lambda_1 - d\lambda_3,$$
$$d\lambda_1 + d\lambda_2 - d\lambda_0 - d\lambda_3\rangle$$
$$=: \langle \alpha, \beta, \gamma\rangle.$$

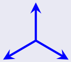
$$\mathcal{P}_2\Lambda^1(T^3)$$
$$= \mathcal{P}_2\Lambda^0(T^3) \otimes \mathcal{P}_0\Lambda^1(T^3)$$
$$= \langle \lambda_0^2\alpha, \lambda_0^2\beta, \lambda_0^2\gamma,$$
$$\lambda_1^2\alpha, \lambda_1^2\beta, \lambda_1^2\gamma,$$
$$\lambda_2^2\alpha, \lambda_2^2\beta, \lambda_2^2\gamma,$$
$$\lambda_3^2\alpha, \lambda_3^2\beta, \lambda_3^2\gamma,$$
$$\lambda_0\lambda_1\alpha, \lambda_0\lambda_1\beta, \lambda_0\lambda_1\gamma,$$
$$\lambda_0\lambda_2\alpha, \lambda_0\lambda_2\beta, \lambda_0\lambda_2\gamma,$$
$$\lambda_0\lambda_3\alpha, \lambda_0\lambda_3\beta, \lambda_0\lambda_3\gamma,$$
$$\lambda_1\lambda_2\alpha, \lambda_1\lambda_2\beta, \lambda_1\lambda_2\gamma,$$
$$\lambda_1\lambda_3\alpha, \lambda_1\lambda_3\beta, \lambda_1\lambda_3\gamma,$$
$$\lambda_2\lambda_3\alpha, \lambda_2\lambda_3\beta, \lambda_2\lambda_3\gamma\rangle.$$

Representations of $\mathbb{Z}/3$

- The 1D representation $\mathbf{1}$ where $\mathbb{Z}/3$ acts trivially.
- The 2D representation $\mathbf{2}$ where $\mathbb{Z}/3$ acts by 120° rotations.
- The 3D representation $\mathbf{3}$ where $\mathbb{Z}/3$ acts by permuting the coordinates.
  - $\mathbf{3} \cong \mathbf{1} \oplus \mathbf{2}$ because $\langle(1,1,1)\rangle$ is an invariant subspace.

Invariant bases

$\mathbf{1}$ and $\mathbf{3}$ have symmetry-invariant bases, but $\mathbf{2}$ does not.



Proposition

A representation $V \cong m\mathbf{1} \oplus n\mathbf{2}$ has a $\mathbb{Z}/3$-invariant basis up to sign if and only if $m \geq n$.

References

Martin Licht.
Symmetry and invariant bases in finite element exterior calculus.
https://arxiv.org/abs/1912.11002.

Yakov Berchenko-Kogan.
Symmetric bases for finite element exterior calculus spaces.
https://arxiv.org/abs/2112.06065.

Douglas N. Arnold, Richard S. Falk, and Ragnar Winther.
Finite element exterior calculus, homological techniques, and applications.
Acta Numer., 15:1–155, 2006.

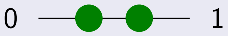
D. N. Arnold and A. Logg.
Periodic Table of the Finite Elements.
SIAM News, 47(9), 2014.

Martin Licht.
On basis constructions in finite element exterior calculus.
Adv. Comput. Math., 48(2), 2022.

Yakov Berchenko-Kogan.
Duality in finite element exterior calculus and Hodge duality on the sphere.
Found. Comput. Math., 21(5):1153–1180, 2021.

Duality

Previously...



$$\langle \lambda_0\, ds, \lambda_1\, ds \rangle \cong \langle \lambda_0^2 \lambda_1, \lambda_1^2 \lambda_0 \rangle$$
$$\mathcal{P}_1\Lambda^1(T^1) \cong \mathring{\mathcal{P}}_3\Lambda^0(T^1)$$

FEEC Duality (Arnold, Falk, and Winther, 2006)

$$\mathcal{P}_r\Lambda^k(T^n) \cong \mathring{\mathcal{P}}^-_{r+k+1}\Lambda^{n-k}(T^n),$$
$$\mathcal{P}^-_r\Lambda^k(T^n) \cong \mathring{\mathcal{P}}_{r+k}\Lambda^{n-k}(T^n).$$

An explicit map (Licht, 2018)

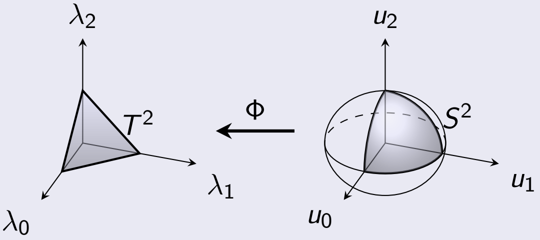
$$\mathcal{P}_1\Lambda^1(T^2) \to \mathring{\mathcal{P}}^-_3\Lambda^1(T^2), \qquad \mathcal{P}^-_1\Lambda^1(T^2) \to \mathring{\mathcal{P}}_2\Lambda^1(T^2),$$
$$\lambda_1\, d\lambda_1 \mapsto \lambda_0\lambda_1^2\, d\lambda_2 - \lambda_1^2\lambda_2\, d\lambda_0, \quad \lambda_0\, d\lambda_1 - \lambda_1\, d\lambda_0 \mapsto \lambda_0\lambda_1\, d\lambda_2.$$

The Hodge star (YBK, 2019)

The two maps are the same; have formula using Hodge star on S^n.

The sphere

Change of coordinates $\lambda_i = u_i^2, \quad d\lambda_i = 2u_i\, du_i$



The duality map

1. Change coordinates to the sphere $\Phi^*\colon \Lambda^k(T^n) \to \Lambda^k(S^n)$.
2. Apply the Hodge star on the sphere.
3. Multiply by the bubble function $u_N := u_0 \cdots u_n$.
4. Change coordinates back to the simplex.

$$(\Phi^*)^{-1} \circ u_N {*_{S^n}} \circ \Phi^*$$

Examples

$a = \lambda_1\, d\lambda_1 \in \mathcal{P}_1\Lambda^1(T^2)$

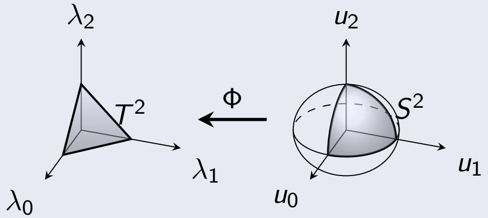
1. $\alpha = \Phi^* a = 2u_1^3\, du_1 \in \mathcal{P}_3\Lambda^1(S^2)$.
2. $*_{S^2}\alpha = 2u_0u_1^3\, du_2 - 2u_1^3u_2\, du_0 \in \mathcal{P}_4^-\Lambda^1(S^2)$.
3. $\beta = u_0u_1u_2(*_{S^n}\alpha) = 2u_0^2u_1^4u_2\, du_2 - 2u_0u_1^4u_2^2\, du_0 \in \mathring{\mathring{\mathcal{P}}}_7^-\Lambda^1(S^2)$.
4. $b = (\Phi^*)^{-1}a = \lambda_0\lambda_1^2\, d\lambda_2 - \lambda_1^2\lambda_2\, d\lambda_0 \in \mathring{\mathcal{P}}_3^-\Lambda^1(T^2)$.

$a = \lambda_0\, d\lambda_1 - \lambda_1\, d\lambda_0 \in \mathcal{P}_1^-\Lambda^1(T^2)$

1. $\alpha = \Phi^* a = 2u_0^2u_1\, du_1 - 2u_0u_1^2\, du_0 \in \mathcal{P}_3^-\Lambda^1(S^2)$.
2. $$\begin{aligned} *_{S^2}\alpha &= 2((u_0^3u_1 + u_0u_1^3)du_2 - u_0^2u_1u_2\, du_0 - u_0u_1^2u_2\, du_1) \\ &= 2u_0u_1(u_0^2 + u_1^2 + u_2^2)\, du_2 - u_0u_1u_2\, d(u_0^2 + u_1^2 + u_2^2) \\ &= 2u_0u_1\, du_2 \in \mathcal{P}_2\Lambda^1(S^2). \end{aligned}$$
3. $\beta = u_0u_1u_2(*_{S^2}\alpha) = 2u_0^2u_1^2u_2\, du_2 \in \mathring{\mathring{\mathcal{P}}}_5\Lambda^1(S^2)$.
4. $b = (\Phi^*)^{-1}\beta = \lambda_0\lambda_1\, d\lambda_2 \in \mathring{\mathcal{P}}_2\Lambda^1(T^2)$.

Polynomial forms on the simplex and the sphere

Change of coordinates $\quad \lambda_i = u_i^2, \quad d\lambda_i = 2u_i\,du_i$



Theorem

The map $\Phi^*\colon \Lambda^k(T^n) \to \Lambda^k(S^n)$ gives isomorphisms:

$$\mathcal{P}_r\Lambda^k(T^n) \xrightarrow{\cong} \mathcal{P}_{2r+k}\Lambda^k_e(S^n),$$

$$\mathcal{P}_r^-\Lambda^k(T^n) \xrightarrow{\cong} \mathcal{P}_{2r+k}^-\Lambda^k_e(S^n),$$

$$\mathring{\mathcal{P}}_r\Lambda^k(T^n) \xrightarrow{\cong} \mathring{\mathring{\mathcal{P}}}_{2r+k}\Lambda^k_e(S^n),$$

$$\mathring{\mathcal{P}}_r^-\Lambda^k(T^n) \xrightarrow{\cong} \mathring{\mathring{\mathcal{P}}}{}_{2r+k}^-\Lambda^k_e(S^n),$$

$\mathcal{P}_r\Lambda^k(T^n) \cong \mathcal{P}_{2r+k}\Lambda^k_e(S^n)$

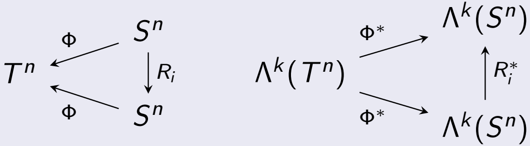
Example

$$\lambda_0\lambda_1^2\, d\lambda_2 \wedge d\lambda_3 \mapsto u_0^2u_1^4(2u_2\, du_2) \wedge (2u_3\, du_3)$$
$$= 4u_0^2u_1^4u_2u_3\, du_2 \wedge du_3$$
$$\mathcal{P}_3\Lambda^2(T^3) \to \mathcal{P}_8\Lambda^2_e(S^3).$$

Definition

- A form is even if it is invariant under all coordinate reflections.
  - e.g. $R_2\colon (u_0, u_1, u_2, u_3) \mapsto (u_0, u_1, -u_2, u_3)$.
- The space of such forms is denoted $\Lambda^k_e(S^n)$.

The image of Φ^* is even



$\mathcal{P}_r^-\Lambda^k(T^n) \cong \mathcal{P}_{2r+k}^-\Lambda_e^k(S^n)$

A new definition of $\mathcal{P}_r^-$

- Let X denote the radial vector field
$$X = (\lambda_0, \dots, \lambda_n) = \lambda_0 \frac{\partial}{\partial\lambda_0} + \cdots + \lambda_n \frac{\partial}{\partial\lambda_n}$$
- Let
$$\mathcal{P}_r^-\Lambda^k(\mathbb{R}^{n+1}) := i_X \mathcal{P}_{r-1}\Lambda^{k+1}(\mathbb{R}^{n+1}).$$
- Let $\mathcal{P}_r^-\Lambda^k(T^n)$ and $\mathcal{P}_r^-\Lambda^k(S^n)$ denote the restrictions of $\mathcal{P}_r^-\Lambda^k(\mathbb{R}^{n+1})$ to T^n and S^n, respectively.

Φ^* sends $\mathcal{P}^-$ to $\mathcal{P}^-$

- View $\Phi\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ with same formula
$$(\lambda_0, \dots, \lambda_n) = \Phi(u_0, \dots, u_n) = (u_0^2, \dots, u_n^2)$$
- Key fact: $\Phi_* X = 2X$.
$$\sum_{i=0}^n u_i \frac{\partial}{\partial u_i} = \sum_{i=0}^n u_i \frac{\partial\lambda_i}{\partial u_i}\frac{\partial}{\partial\lambda_i} = \sum_{i=0}^n u_i(2u_i)\frac{\partial}{\partial\lambda_i} = 2\sum_{i=0}^n \lambda_i \frac{\partial}{\partial\lambda_i}.$$

$\mathring{\mathcal{P}}_r\Lambda^k(T^n) \cong \mathring{\mathring{\mathcal{P}}}_{2r+k}\Lambda^k_e(S^n)$

Two notions of “vanishing trace”

Let Ω be a domain with boundary $\partial\Omega$ and let $\alpha \in \Lambda^k(\Omega)$.

1. $i^*\alpha = 0$, where $i\colon \partial\Omega \hookrightarrow \Omega$ is the inclusion.
   - $\alpha(X_1, \dots, X_k) = 0$ for any vectors $X_1, \dots, X_k$ tangent to $\partial\Omega$.
   - $\alpha \in \mathring{\mathcal{P}}_r\Lambda^k(T^n)$ vanishes if we set $\lambda_i = 0$ and $d\lambda_i = 0$.
2. Evaluated at any point $x \in \partial\Omega$, $\alpha_x \in \Lambda^k T^*_x\Omega$ vanishes.
   - $\alpha(X_1, \dots, X_k) = 0$ for any vectors $X_1, \dots, X_k$ based at $\partial\Omega$.
   - The coefficients of α vanish on $\partial\Omega$.
   - $\alpha \in \mathring{\mathring{\mathcal{P}}}_{2r+k}\Lambda^k_e(S^n)$ vanishes if we set $u_i = 0$.

Φ^* sends $\mathring{\mathcal{P}}$ to $\mathring{\mathring{\mathcal{P}}}$

- Let $S_i^{n-1} = S^n \cap \{u_i = 0\}$ and $T_i^{n-1} = T^n \cap \{\lambda_i = 0\}$.
- $D\Phi$ maps vectors tangent to S_i^{n-1} to vectors tangent to T_i^{n-1}.
- $D\Phi$ maps vectors normal to S_i^{n-1} to zero. ($\frac{\partial u_i^2}{\partial u_i} = 0$ on S_i^{n-1}.)

Recap

Theorem

The map $\Phi^*\colon \Lambda^k(T^n) \to \Lambda^k(S^n)$ *gives isomorphisms:*

$$\mathcal{P}_r\Lambda^k(T^n) \xrightarrow{\cong} \mathcal{P}_{2r+k}\Lambda^k_e(S^n),$$
$$\mathcal{P}_r^-\Lambda^k(T^n) \xrightarrow{\cong} \mathcal{P}_{2r+k}^-\Lambda^k_e(S^n),$$
$$\mathring{\mathcal{P}}_r\Lambda^k(T^n) \xrightarrow{\cong} \mathring{\mathring{\mathcal{P}}}_{2r+k}\Lambda^k_e(S^n),$$
$$\mathring{\mathcal{P}}_r^-\Lambda^k(T^n) \xrightarrow{\cong} \mathring{\mathring{\mathcal{P}}}{}_{2r+k}^-\Lambda^k_e(S^n),$$

The duality map

$$(\Phi^*)^{-1} \circ u_{N^*S^n} \circ \Phi^*\colon \quad \begin{aligned} \mathcal{P}_r\Lambda^k(T^n) &\cong \mathring{\mathcal{P}}_{r+k+1}^-\Lambda^{n-k}(T^n), \\ \mathcal{P}_r^-\Lambda^k(T^n) &\cong \mathring{\mathcal{P}}_{r+k}\Lambda^{n-k}(T^n). \end{aligned}$$

$$u_{N^*S^n}\colon \quad \begin{aligned} \mathcal{P}_{2r+k}\Lambda^k_e(S^n) &\cong \mathring{\mathring{\mathcal{P}}}{}_{2r+n+k+2}^-\Lambda^{n-k}_e(S^n), \\ \mathcal{P}_{2r+k}^-\Lambda^k_e(S^n) &\cong \mathring{\mathring{\mathcal{P}}}_{2r+n+k}\Lambda^{n-k}_e(S^n). \end{aligned}$$

The Hodge star on the sphere

Proposition

$$*_{S^n}: \quad \begin{aligned} \mathcal{P}_s\Lambda^k(S^n) &\cong \mathcal{P}^-_{s+1}\Lambda^{n-k}(S^n), \\ \mathcal{P}^-_s\Lambda^k(S^n) &\cong \mathcal{P}_{s-1}\Lambda^{n-k}(S^n). \end{aligned}$$

Example

$$\begin{aligned} \alpha &= u_1^3\,du_1 \in \mathcal{P}_3\Lambda^1(S^2). \\ *_{S^2}\,\alpha &= -i_X(*_{\mathbb{R}^3}\alpha) \\ &= i_X(u_1^3\,du_0 \wedge du_2) \\ &= u_1^3 i_X(du_0)\,du_2 - u_1^3 i_X(du_2)\,du_0 \\ &= u_1^3 u_0\,du_2 - u_1^3 u_2\,du_0 \in \mathcal{P}^-_4\Lambda^1(S^2) \\ *_{S^2}\,(*_{S^2}\alpha) &= u_1^3 u_0(u_1\,du_0 - u_0\,du_1) - u_1^3 u_2(u_2\,du_1 - u_1\,du_2) \\ &= -u_1^3(u_0^2 + u_1^2 + u_2^2)\,du_1 + \tfrac{1}{2}u_1^4\,d(u_0^2 + u_1^2 + u_2^2) \\ &= -\alpha \in \mathcal{P}_3\Lambda^1(S^2). \end{aligned}$$

Multiplication by the bubble function

Proposition

$$u_N = u_0 \cdots u_n : \qquad \begin{aligned} \mathcal{P}_s\Lambda^k(S^n) &\cong \mathring{\mathcal{P}}_{s+n+1}\Lambda^k(S^n), \\ \mathcal{P}_s^-\Lambda^k(S^n) &\cong \mathring{\mathcal{P}}_{s+n+1}^-\Lambda^k(S^n). \end{aligned}$$

Counterexamples

- $u_N = i_X(u_1 \cdots u_n \, du_0) \in \mathring{\mathcal{P}}_{n+1}^-\Lambda^0(S^n)$.
  - But $1 \notin \mathcal{P}_0^-\Lambda^0(S^n)$.
- $u_N \operatorname{vol}_{S^n} = u_1 \cdots u_n \, du_1 \wedge \cdots \wedge du_n \in \mathring{\mathcal{P}}_n\Lambda^n(S^n)$.
  - But $\operatorname{vol}_{S^n} \notin \mathcal{P}_{-1}\Lambda^n(S^n)$.

It's okay

- These are the only counterexamples.
- Proposition still holds if we add $\operatorname{span}\{1\}$ or $\operatorname{span}\{\operatorname{vol}_{S^n}\}$ to the left-hand side when necessary.

Recap

The duality map

$$u_{N*S^n}: \quad \begin{aligned} \mathcal{P}_{2r+k}\Lambda^k_e(S^n) &\cong \mathring{\mathring{\mathcal{P}}}^-_{2r+n+k+2}\Lambda^{n-k}_e(S^n), \\ \mathcal{P}^-_{2r+k}\Lambda^k_e(S^n) &\cong \mathring{\mathring{\mathcal{P}}}_{2r+n+k}\Lambda^{n-k}_e(S^n). \end{aligned}$$

$$(\Phi^*)^{-1} \circ u_{N*S^n} \circ \Phi^*: \quad \begin{aligned} \mathcal{P}_r\Lambda^k(T^n) &\cong \mathring{\mathcal{P}}^-_{r+k+1}\Lambda^{n-k}(T^n), \\ \mathcal{P}^-_r\Lambda^k(T^n) &\cong \mathring{\mathcal{P}}_{r+k}\Lambda^{n-k}(T^n). \end{aligned}$$

Exception

- $\mathrm{vol}_{T^n} \in \mathring{\mathcal{P}}_0\Lambda^n(T^n)$ but $1 \notin \mathcal{P}^-_0\Lambda^0(T^n)$.

A special case or a new definition?

Exception

- $\mathrm{vol}_{T^n} \in \mathring{\mathcal{P}}_0\Lambda^n(T^n)$ but $1 \notin \mathcal{P}_0^-\Lambda^0(T^n)$.

Definition

$$\mathcal{P}_r^-\Lambda^k(\mathbb{R}^{n+1}) := i_X\mathcal{P}_{r-1}\Lambda^{k+1}(\mathbb{R}^{n+1}).$$

Definition

$$\hat{\mathcal{P}}_r^-\Lambda^k(\mathbb{R}^{n+1}) := \{\alpha \in \mathcal{P}_r\Lambda^k(\mathbb{R}^{n+1}) \mid i_X\alpha = 0\}$$

Proposition

Restricting to T^n,

$$\hat{\mathcal{P}}_r^-\Lambda^k(T^n) = \mathcal{P}_r^-\Lambda^k(T^n)$$

except $\hat{\mathcal{P}}_0^-\Lambda^0(T^n) = \mathrm{span}\{1\}$.

While we're at it

Exceptions

- $u_N = i_X(u_1 \cdots u_n \, du_0) \in \mathring{\mathring{\mathcal{P}}}^-_{n+1}\Lambda^0(S^n)$.
  - But $1 \notin \mathcal{P}_0^-\Lambda^0(S^n)$.
- $u_N \operatorname{vol}_{S^n} = u_1 \cdots u_n \, du_1 \wedge \cdots \wedge du_n \in \mathring{\mathring{\mathcal{P}}}_n\Lambda^n(S^n)$.
  - But $\operatorname{vol}_{S^n} \notin \mathcal{P}_{-1}\Lambda^n(S^n)$.

The volume form

$$\operatorname{vol}_{S^n} = u_0^{-1} du_1 \wedge \cdots du_n.$$

Perhaps it should be in $\mathcal{P}_{-1}\Lambda^n(S^n)$ after all?

A vague parallel?

The cohomology of smooth closed manifolds

- $\mathcal{H}^k(M) := \{\alpha \in \Lambda^k(M) \mid d\alpha = 0 \text{ and } d(*\alpha) = 0\}$.
  - $\mathcal{H}^k(M) \cong H^k(M)$.
- $\alpha \in \mathcal{H}^k(M) \Leftrightarrow *\alpha \in \mathcal{H}^{n-k}(M)$ so $H^k(M) \cong H^{n-k}(M)$.

The cohomology of smooth manifolds with boundary $i\colon \partial M \hookrightarrow M$

- $\mathring{\mathcal{H}}^k(M) := \{\alpha \in \Lambda^k(M) \mid d\alpha = d(*\alpha) = 0 \text{ and } i^*\alpha = 0\}$.
  - $\mathring{\mathcal{H}}^k(M) \cong H^k(M, \partial M)$.
- $\dot{\mathcal{H}}^k(M) := \{\alpha \in \Lambda^k(M) \mid d\alpha = d(*\alpha) = 0 \text{ and } i^*(*\alpha) = 0\}$.
  - $\dot{\mathcal{H}}^k(M) \cong H^k(M)$.
- $\alpha \in \dot{\mathcal{H}}^k(M) \Leftrightarrow *\alpha \in \mathring{\mathcal{H}}^{n-k}(M)$ so $H^k(M) \cong H^{n-k}(M, \partial M)$.
- $H^k(M) := \{\alpha \in \Lambda^k(M) \mid d\alpha = 0\} / \{\alpha \in \Lambda^k(M) \mid \alpha = d\beta\}$.
  - No boundary conditions!
- Duality between $H^k(M)$ and $\mathring{\mathcal{H}}^{n-k}(M)$?

A vague connection?

The cohomology of smooth manifolds with boundary $i\colon \partial M \hookrightarrow M$

- $\mathring{\mathcal{H}}^k(M) := \{\alpha \in \Lambda^k(M) \mid d\alpha = d(*\alpha) = 0 \text{ and } i^*\alpha = 0\}$.
  - $\mathring{\mathcal{H}}^k(M) \cong H^k(M, \partial M)$.
- $\dot{\mathcal{H}}^k(M) := \{\alpha \in \Lambda^k(M) \mid d\alpha = d(*\alpha) = 0 \text{ and } i^*(*\alpha) = 0\}$.
  - $\dot{\mathcal{H}}^k(M) \cong H^k(M)$.
- $\alpha \in \dot{\mathcal{H}}^k(M) \Leftrightarrow *\alpha \in \mathring{\mathcal{H}}^{n-k}(M)$ so $H^k(M) \cong H^{n-k}(M, \partial M)$.
- $H^k(M) := \{\alpha \in \Lambda^k(M) \mid d\alpha = 0\}/\{\alpha \in \Lambda^k(M) \mid \alpha = d\beta\}$.
  - No boundary conditions!
- Duality between $H^k(M)$ and $\mathring{\mathcal{H}}^{n-k}(M)$?

Duality

$$\mathcal{P}_r\Lambda^k(T^n) \cong \mathring{\mathcal{P}}^-_{r+k+1}\Lambda^{n-k}(T^n),$$
$$\mathcal{P}^-_r\Lambda^k(T^n) \cong \mathring{\mathcal{P}}_{r+k}\Lambda^{n-k}(T^n).$$

Thank you

Martin Licht.
Symmetry and invariant bases in finite element exterior calculus.
https://arxiv.org/abs/1912.11002.

Yakov Berchenko-Kogan.
Symmetric bases for finite element exterior calculus spaces.
https://arxiv.org/abs/2112.06065.

Douglas N. Arnold, Richard S. Falk, and Ragnar Winther.
Finite element exterior calculus, homological techniques, and applications.
Acta Numer., 15:1–155, 2006.

D. N. Arnold and A. Logg.
Periodic Table of the Finite Elements.
SIAM News, 47(9), 2014.

Martin Licht.
On basis constructions in finite element exterior calculus.
Adv. Comput. Math., 48(2), 2022.

Yakov Berchenko-Kogan.
Duality in finite element exterior calculus and Hodge duality on the sphere.
Found. Comput. Math., 21(5):1153–1180, 2021.