Symmetry in Finite Element Exterior Calculus

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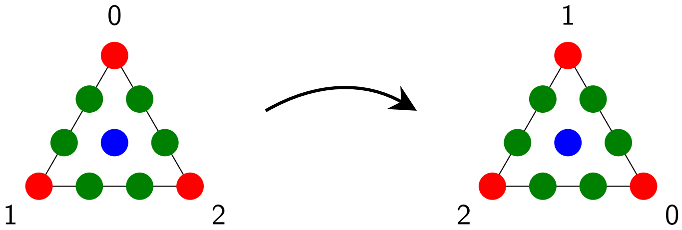
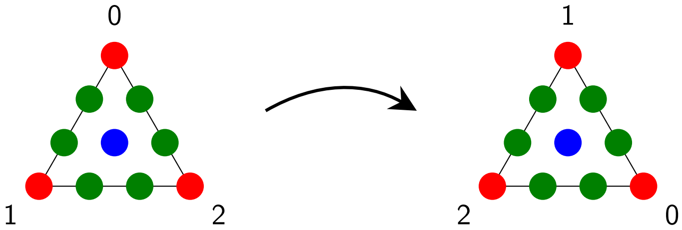
Symmetry of Scalar Elements



$$\mathcal{P}_3\Lambda^0(T^2) = \left\langle \lambda_0^3, \lambda_1^3, \lambda_2^3, \lambda_1^2\lambda_2, \lambda_2^2\lambda_1, \lambda_2^2\lambda_0, \lambda_0^2\lambda_2, \lambda_0^2\lambda_1, \lambda_1^2\lambda_0, \lambda_0\lambda_1\lambda_2 \right\rangle .$$



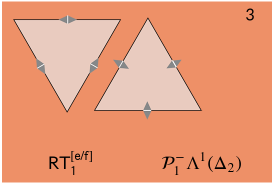
$$\mathcal{P}_3\Lambda^0(T^2) = \left\langle \lambda_0^3, \lambda_1^3, \lambda_2^3, \lambda_1^2\lambda_2, \lambda_2^2\lambda_1, \lambda_2^2\lambda_0, \lambda_0^2\lambda_2, \lambda_0^2\lambda_1, \lambda_1^2\lambda_0, \lambda_0\lambda_1\lambda_2 \right\rangle .$$

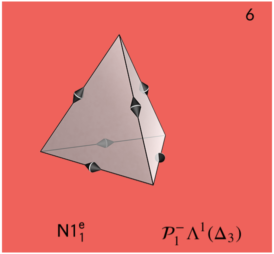
- When computing matrix of, e.g., $a(u,v) = \int_{T^2} \nabla u \cdot \nabla v$, can exploit sixfold symmetry of T^2 to compute fewer entries.

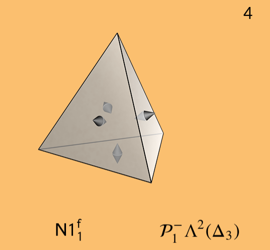
$$\begin{aligned} a\left(\lambda_0^3, \lambda_1^2\lambda_2\right) &= a\left(\lambda_0^3, \lambda_2^2\lambda_1\right) \\ = a\left(\lambda_1^3, \lambda_2^2\lambda_0\right) &= a\left(\lambda_1^3, \lambda_0^2\lambda_2\right) \\ = a\left(\lambda_2^3, \lambda_0^2\lambda_1\right) &= a\left(\lambda_2^3, \lambda_1^2\lambda_0\right) \end{aligned}$$

Symmetry of Vector Elements

Whitney Elements





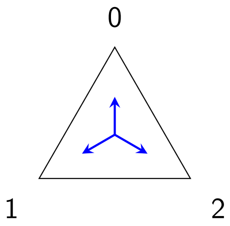


$$\begin{aligned}\langle \lambda_1\, d\lambda_2 - \lambda_2\, d\lambda_1,\\ \lambda_2\, d\lambda_0 - \lambda_0\, d\lambda_2,\\ \lambda_0\, d\lambda_1 - \lambda_1\, d\lambda_0 \rangle.\end{aligned}$$

$$\begin{aligned}\langle \lambda_1\, d\lambda_2 - \lambda_2\, d\lambda_1,\\ \lambda_2\, d\lambda_0 - \lambda_0\, d\lambda_2,\\ \lambda_0\, d\lambda_1 - \lambda_1\, d\lambda_0,\\ \lambda_0\, d\lambda_3 - \lambda_3\, d\lambda_0,\\ \lambda_1\, d\lambda_3 - \lambda_3\, d\lambda_1,\\ \lambda_2\, d\lambda_3 - \lambda_3\, d\lambda_2 \rangle.\end{aligned}$$

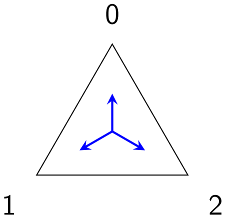
$$\begin{aligned}\langle \quad & \lambda_1\, d\lambda_2 \wedge d\lambda_3\\ &+ \lambda_2\, d\lambda_3 \wedge d\lambda_1\\ &+ \lambda_3\, d\lambda_1 \wedge d\lambda_2,\\ &\ldots,\\ &\lambda_0\, d\lambda_1 \wedge d\lambda_2\\ &+ \lambda_1\, d\lambda_2 \wedge d\lambda_0\\ &+ \lambda_2\, d\lambda_0 \wedge d\lambda_1 \rangle\end{aligned}$$

Geometric symmetry $\Rightarrow$ basis symmetry (up to sign).



$\mathcal{P}_0\Lambda^1(T^2)$
$= \langle d\lambda_0, d\lambda_1, d\lambda_2 \rangle,$

$d\lambda_0 + d\lambda_1 + d\lambda_2 = 0$



$$\mathcal{P}_0\Lambda^1(T^2) = \langle d\lambda_0, d\lambda_1, d\lambda_2 \rangle,$$

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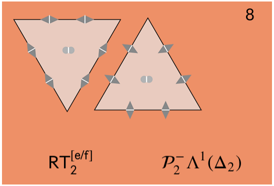
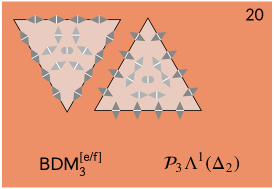


$$\begin{aligned}
\langle &\lambda_1^2\, d\lambda_2 - \lambda_1\lambda_2\, d\lambda_1, \\
&\lambda_2^2\, d\lambda_1 - \lambda_1\lambda_2\, d\lambda_2, \\
&\ldots, \\
&\lambda_0^2\, d\lambda_1 - \lambda_0\lambda_1\, d\lambda_0, \\
&\lambda_1^2\, d\lambda_0 - \lambda_0\lambda_1\, d\lambda_1, \\
&\lambda_0\lambda_1\, d\lambda_2 - \lambda_0\lambda_2\, d\lambda_1, \\
&\lambda_1\lambda_2\, d\lambda_0 - \lambda_0\lambda_1\, d\lambda_2, \\
&\lambda_0\lambda_2\, d\lambda_1 - \lambda_1\lambda_2\, d\lambda_0 \rangle.
\end{aligned}$$



$$\begin{aligned}
\langle &\ldots, \\
&\ldots, \\
&\lambda_0\lambda_1\lambda_2\, d\lambda_0, \\
&\lambda_0\lambda_1\lambda_2\, d\lambda_1, \\
&\lambda_0\lambda_1\lambda_2\, d\lambda_2 \rangle.
\end{aligned}$$

Results

Theorem (if: Licht, 2019; only if: YBK, 2021)

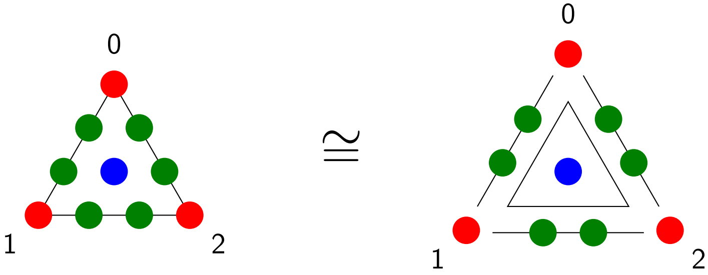
The following spaces have symmetry-invariant bases up to sign if and only if the corresponding condition holds.

$$\begin{array}{lcl} \mathcal{P}_r\Lambda^1(T^2) & \textit{if and only if} & r \notin 3\mathbb{N}_0, \\ \mathcal{P}_r^-\Lambda^1(T^2) & \textit{if and only if} & r \notin 3\mathbb{N}_0 + 2. \end{array}$$

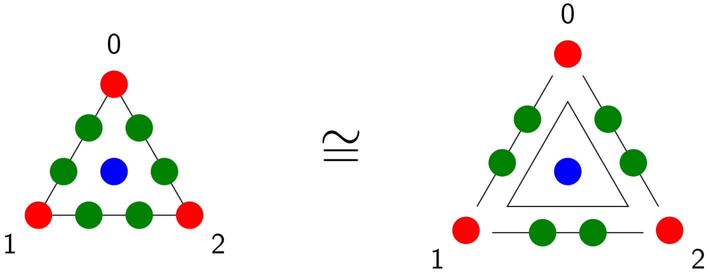
Theorem (YBK, 2021)

The following spaces have symmetry-invariant bases up to sign if and only if the corresponding condition holds.

$$\begin{array}{lcl} \mathcal{P}_r\Lambda^1(T^3) & \textit{always}, & \\ \mathcal{P}_r^-\Lambda^1(T^3) & \textit{if and only if} & r \notin 3\mathbb{N}_0 + 2, \\ \mathcal{P}_r\Lambda^2(T^3) & \textit{always}, & \\ \mathcal{P}_r^-\Lambda^2(T^3) & \textit{always}. & \end{array}$$

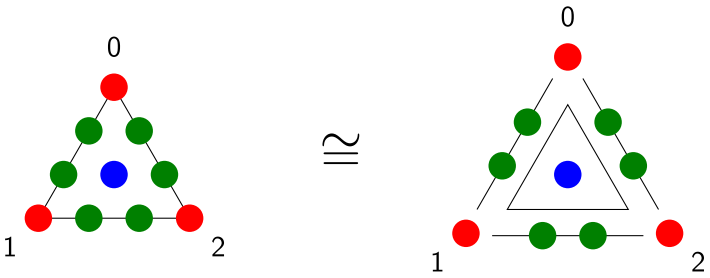


$$\mathcal{P}_3\Lambda^0(T^2) \cong 3\mathring{\mathcal{P}}_3\Lambda^0(T^0) \oplus 3\mathring{\mathcal{P}}_3\Lambda^0(T^1) \oplus \mathring{\mathcal{P}}_3\Lambda^0(T^2)$$



$$\mathcal{P}_3\Lambda^0(T^2) \cong 3\mathring{\mathcal{P}}_3\Lambda^0(T^0) \oplus 3\mathring{\mathcal{P}}_3\Lambda^0(T^1) \oplus \mathring{\mathcal{P}}_3\Lambda^0(T^2)$$

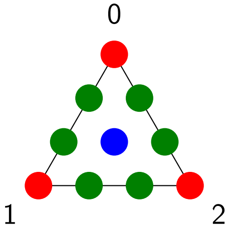
$$\langle \lambda_0^3 \rangle \oplus \langle \lambda_1^3 \rangle \oplus \langle \lambda_2^3 \rangle$$
$$\oplus \langle \lambda_1^2\lambda_2, \lambda_2^2\lambda_1 \rangle \oplus \langle \lambda_2^2\lambda_0, \lambda_0^2\lambda_2 \rangle \oplus \langle \lambda_0^2\lambda_1, \lambda_1^2\lambda_0 \rangle \oplus \langle \lambda_0\lambda_1\lambda_2 \rangle$$



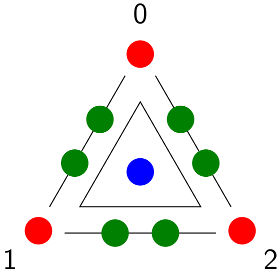
$$\mathcal{P}_3\Lambda^0(T^2) \cong 3\mathring{\mathcal{P}}_3\Lambda^0(T^0) \oplus 3\mathring{\mathcal{P}}_3\Lambda^0(T^1) \oplus \mathring{\mathcal{P}}_3\Lambda^0(T^2)$$

$$\begin{aligned} & \langle\lambda_0^3\rangle \oplus \langle\lambda_1^3\rangle \oplus \langle\lambda_2^3\rangle \\ & \quad \oplus \langle\lambda_1^2\lambda_2, \lambda_2^2\lambda_1\rangle \oplus \langle\lambda_2^2\lambda_0, \lambda_0^2\lambda_2\rangle \oplus \langle\lambda_0^2\lambda_1, \lambda_1^2\lambda_0\rangle \oplus \langle\lambda_0\lambda_1\lambda_2\rangle \\ \cong\ & \langle\lambda_0^2\rangle \oplus \langle\lambda_1^2\rangle \oplus \langle\lambda_2^2\rangle \\ & \quad \oplus \langle\lambda_1\, ds, \lambda_2\, ds\rangle \oplus \langle\lambda_0\, ds, \lambda_2\, ds\rangle \oplus \langle\lambda_0\, ds, \lambda_1\, ds\rangle \oplus \langle 1\, dA\rangle \end{aligned}$$

Methods
Recursion

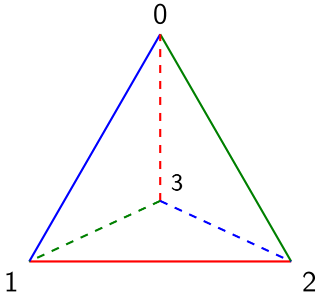


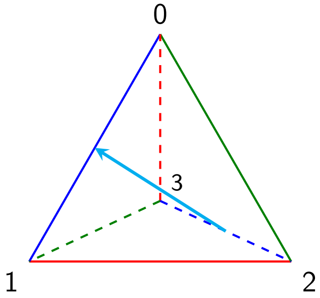
$\cong$



$$\begin{aligned}\mathcal{P}_3\Lambda^0(T^2) &\cong 3\mathring{\mathcal{P}}_3\Lambda^0(T^0) \oplus 3\mathring{\mathcal{P}}_3\Lambda^0(T^1) \oplus \mathring{\mathcal{P}}_3\Lambda^0(T^2)\\ &\cong 3\mathcal{P}_2\Lambda^0(T^0) \oplus 3\mathcal{P}_1\Lambda^1(T^1) \oplus \mathcal{P}_0\Lambda^2(T^2)\end{aligned}$$

$$\begin{aligned}&\langle\lambda_0^3\rangle \oplus \langle\lambda_1^3\rangle \oplus \langle\lambda_2^3\rangle\\ &\quad \oplus \langle\lambda_1^2\lambda_2, \lambda_2^2\lambda_1\rangle \oplus \langle\lambda_2^2\lambda_0, \lambda_0^2\lambda_2\rangle \oplus \langle\lambda_0^2\lambda_1, \lambda_1^2\lambda_0\rangle \oplus \langle\lambda_0\lambda_1\lambda_2\rangle\\ \cong\ &\langle\lambda_0^2\rangle \oplus \langle\lambda_1^2\rangle \oplus \langle\lambda_2^2\rangle\\ &\quad \oplus \langle\lambda_1\,ds, \lambda_2\,ds\rangle \oplus \langle\lambda_0\,ds, \lambda_2\,ds\rangle \oplus \langle\lambda_0\,ds, \lambda_1\,ds\rangle \oplus \langle 1\,dA\rangle\end{aligned}$$


0
3
1
2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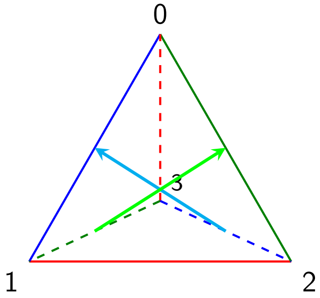

0
3
1
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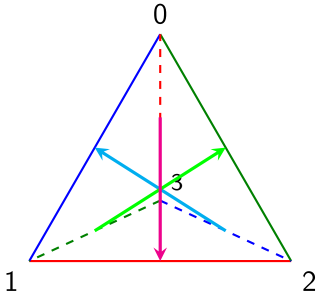
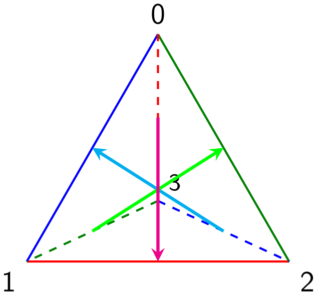
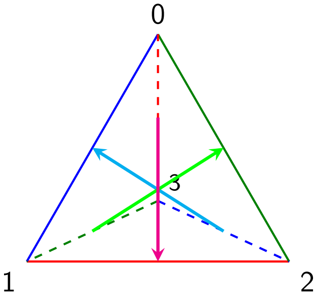
Methods
Tetrahedron Basis





$$
\begin{aligned}
&\mathcal{P}_0\Lambda^1(T^3)\\
&= \langle d\lambda_0 + d\lambda_1 - d\lambda_2 - d\lambda_3,\\
&\quad d\lambda_0 + d\lambda_2 - d\lambda_1 - d\lambda_3,\\
&\quad d\lambda_1 + d\lambda_2 - d\lambda_0 - d\lambda_3\rangle\\
&=: \langle \alpha, \beta, \gamma\rangle.
\end{aligned}
$$



$$\mathcal{P}_0\Lambda^1(T^3)$$
$$= \langle d\lambda_0 + d\lambda_1 - d\lambda_2 - d\lambda_3, \quad d\lambda_0 + d\lambda_2 - d\lambda_1 - d\lambda_3, \quad d\lambda_1 + d\lambda_2 - d\lambda_0 - d\lambda_3\rangle$$
$$=: \langle \alpha, \beta, \gamma\rangle.$$

$$\mathcal{P}_2\Lambda^1(T^3)$$
$$= \mathcal{P}_2\Lambda^0(T^3) \otimes \mathcal{P}_0\Lambda^1(T^3)$$
$$= \langle \lambda_0^2\alpha, \lambda_0^2\beta, \lambda_0^2\gamma,$$
$$\lambda_1^2\alpha, \lambda_1^2\beta, \lambda_1^2\gamma,$$
$$\lambda_2^2\alpha, \lambda_2^2\beta, \lambda_2^2\gamma,$$
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$$\lambda_2\lambda_3\alpha, \lambda_2\lambda_3\beta, \lambda_2\lambda_3\gamma\rangle.$$

Representations of $\mathbb{Z}/3$

- The 1D representation $\mathbf{1}$ where $\mathbb{Z}/3$ acts trivially.
- The 2D representation $\mathbf{2}$ where $\mathbb{Z}/3$ acts by 120° rotations.
- The 3D representation $\mathbf{3}$ where $\mathbb{Z}/3$ acts by permuting the coordinates.
  - $\mathbf{3} \cong \mathbf{1} \oplus \mathbf{2}$ because $\langle(1,1,1)\rangle$ is an invariant subspace.

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Invariant bases

$\mathbf{1}$ and $\mathbf{3}$ have symmetry-invariant bases, but $\mathbf{2}$ does not.

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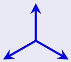

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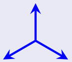

Proposition

A representation $V \cong m\mathbf{1} \oplus n\mathbf{2}$ has a $\mathbb{Z}/3$-invariant basis up to sign if and only if $m \geq n$.

Thank You and References

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