Duality in Finite Element Exterior Calculus and the Hodge Star Operator on the Sphere

Yakov Berchenko-Kogan

Washington University in St. Louis

March 24, 2019

The finite element method

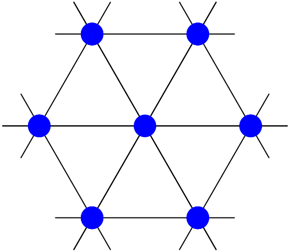


Figure: Degrees of freedom (blue) of piecewise linear functions (left) and piecewise quadratic functions (right).

Geometric decomposition

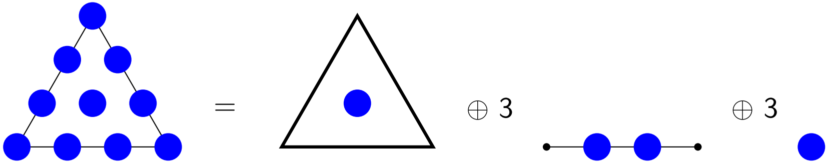


Figure: $\mathcal{P}_3(T^2) = \mathring{\mathcal{P}}_3(T^2) \oplus 3\mathring{\mathcal{P}}_3(T^1) \oplus 3\mathring{\mathcal{P}}_3(T^0)$

$$\begin{aligned}(\mathcal{P}_3(T^2))^* &\cong (\mathring{\mathcal{P}}_3(T^2))^* \oplus 3(\mathring{\mathcal{P}}_3(T^1))^* \oplus 3(\mathring{\mathcal{P}}_3(T^0))^* \\ &\cong \mathcal{P}_0(T^2) \oplus 3\mathcal{P}_1(T^1) \oplus 3\mathcal{P}_2(T^0)\end{aligned}$$

Duality

Definition

- Let $\mathcal{P}$ and $\mathcal{Q}$ be spaces of functions $T^n \to \mathbb{R}$.
    - e.g. $\mathcal{P} = \mathcal{P}_1(T^1)$, $\mathcal{Q} = \mathring{\mathcal{P}}_3(T^1))$
- Consider the pairing
$$(p, q) \mapsto \int_{T^n} pq.$$
- $\mathcal{P}$ and $\mathcal{Q}$ are dual to each other with respect to integration if this pairing is a perfect pairing $\mathcal{P} \times \mathcal{Q} \to \mathbb{R}$.
    - For each nonzero $p \in \mathcal{P}$ there exists a $q \in \mathcal{Q}$ such that $\int_{T^n} pq > 0$, and for each nonzero q there exists such a p.

Problem

Construct a bijection $\mathcal{P} \to \mathcal{Q}$ so that for nonzero $p \mapsto q$ we have

- q only depends on p pointwise, and
- $\int_{T^n} pq > 0$.

Explicit pointwise duality

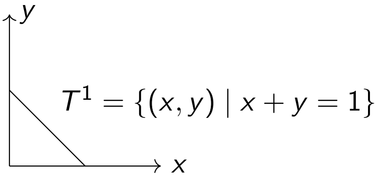


Figure: Barycentric coordinates

Example (Duality between $\mathcal{P}_1(T^1)$ and $\mathring{\mathcal{P}}_3(T^1)$)

For $p \in \mathcal{P}_1(T^1)$, set $q = (xy)p$. Likewise, given q, set $p = \frac{q}{xy}$.

$\mathcal{P}_1(T^1)$	$\mathring{\mathcal{P}}_3(T^1)$	$\int_{T^1} pq$
x	x^2y	$\int_{T^1}(xy)x^2$
y	xy^2	$\int_{T^1}(xy)y^2$

Finite element exterior calculus

Spaces $\mathcal{P}_r\Lambda^k(T^n)$ and $\mathcal{P}_r^-\Lambda^k(T^n)$ of k-forms on T^n with polynomial coefficients of degree at most r.

Special cases

- scalar fields
  - Lagrange
  - Discontinuous Galerkin
- vector fields
  - Brezzi–Douglas–Marini elements
  - Raviart–Thomas elements
  - Nédélec elements

Example

$\mathcal{P}_r\Lambda^1(T^3)$ and $\mathcal{P}_r^-\Lambda^1(T^3)$ are Nédélec $H(\mathrm{curl})$ elements of the 2nd and 1st kinds, respectively.

- See Arnold, Falk, Winther, 2006.

Duality in finite element exterior calculus

Theorem (Arnold, Falk, and Winther)

With respect to the integration pairing

$$(a, b) \mapsto \int_{T^n} a \wedge b$$

- $\mathcal{P}_r\Lambda^k(T^n)$ *is dual to* $\mathring{\mathcal{P}}^-_{r+k+1}\Lambda^{n-k}(T^n)$,
- $\mathcal{P}^-_r\Lambda^k(T^n)$ *is dual to* $\mathring{\mathcal{P}}_{r+k}\Lambda^{n-k}(T^n)$.

Problem

Construct an explicit bijection between these spaces so that for nonzero $a \mapsto b$ we have

- b only depends on a pointwise, and
- $\int_{T^n} a \wedge b > 0$.

A motivating example

Let Ω be an 3-dimensional domain.

- $\Lambda^1(\Omega)$ and $\Lambda^2(\Omega)$ are dual to each other with respect to integration.

Explicit pointwise duality

- Given nonzero $a \in \Lambda^1(\Omega)$, let

$$a = a_x\, dx + a_y\, dy + a_z\, dz.$$

- Define $b \in \Lambda^2(\Omega)$ by

$$b = a_x\, dy \wedge dz + a_y\, dz \wedge dx + a_z\, dx \wedge dy =: *a.$$

- b only depends on a pointwise.
- .

$$\int_\Omega a \wedge b = \int_\Omega (a_x^2 + a_y^2 + a_z^2)\, d\mathrm{vol} = \int_\Omega \|a\|^2\, d\mathrm{vol} > 0,$$

The simplex and the sphere

- T^2 consists of points in the first orthant with $x + y + z = 1$.
- Via the change of coordinates

$$x = u^2, \qquad y = v^2, \qquad z = w^2,$$

we obtain the unit sphere $u^2 + v^2 + w^2 = 1$.

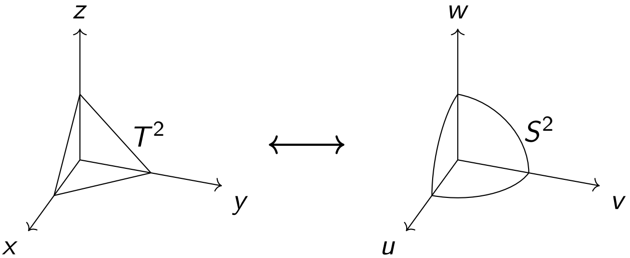


Figure: Change of coordinates

Explicit pointwise duality for finite element exterior calculus

Constructing the dual

1. Start with $a \in \Lambda^k(T^n)$.
2. Change coordinates to obtain $\alpha \in \Lambda^k(S^n)$.
3. Apply the Hodge star to obtain $*_{S^n}\alpha \in \Lambda^{n-k}(S^n)$.
4. Multiply by the coordinate functions to obtain $\beta \in \Lambda^{n-k}(S^n)$.
   - in dimension 2, $\beta = uvw(*_{S^2}\alpha)$.
5. Change coordinates back to obtain $b \in \Lambda^{n-k}(T^n)$.

Theorem (YBK)

- *b depends on a pointwise.*
- $\int_{T^n} a \wedge b > 0$ *for nonzero a.*
- $a \in \mathcal{P}_r\Lambda^k(T^n)$ *iff* $b \in \mathring{\mathcal{P}}^-_{r+k+1}\Lambda^{n-k}(T^n)$.
- $a \in \mathcal{P}^-_r\Lambda^k(T^n)$ *iff* $b \in \mathring{\mathcal{P}}_{r+k}\Lambda^{n-k}(T^n)$.

Example 1

Change of coordinates

$$x = u^2, \qquad y = v^2, \qquad z = w^2,$$
$$dx = 2u\,du, \qquad dy = 2v\,dv, \qquad dz = 2w\,dw.$$

Hodge star on the sphere

$$\nu = u\,du + v\,dv + w\,dw,$$
$$*_{S^2}\alpha = *_{\mathbb{R}^3}(\nu \wedge \alpha).$$

Example

1. $a = y\,dy \in \mathcal{P}_1\Lambda^1(T^2)$.
2. $\alpha = 2v^3\,dv$.
3. $*_{S^2}\alpha = 2uv^3\,dw - 2v^3w\,du$.
4. $\beta = uvw(*_{S^n}\alpha) = 2u^2v^4w\,dw - 2uv^4w^2\,du$.
5. $b = xy^2\,dz - y^2z\,dx \in \mathring{\mathcal{P}}_3^-\Lambda^1(T^2)$.

Example 2

Example

1. $a = x\,dy - y\,dx \in \mathcal{P}_1^- \Lambda^1(T^2)$.
2. $\alpha = 2u^2v\,dv - 2uv^2\,du$.
3. $$\begin{aligned} *_{S^2}\alpha &= 2((u^3v + uv^3)dw - u^2vw\,du - uv^2w\,dv) \\ &= 2uv(u^2+v^2+w^2)\,dw - uvw\,d(u^2+v^2+w^2) \\ &= 2uv\,dw. \end{aligned}$$
4. $\beta = uvw(*_{S^2}\alpha) = 2u^2v^2w\,dw$.
5. $b = xy\,dz \in \mathring{\mathcal{P}}_2\Lambda^1(T^2)$.

Integration via u-substitution

$$\int_{T^2} a \wedge b = \int_{S^2_{>0}} \alpha \wedge \beta = \int_{S^2_{>0}} uvw(\alpha \wedge *_{S^2}\alpha)$$

$$= \int_{S^2_{>0}} uvw\,\|\alpha\|^2 \; d\mathrm{Area} > 0$$

Correspondence between forms on T^n and forms on S^n

Change of coordinates

$$x = u^2, \qquad y = v^2, \qquad z = w^2,$$
$$dx = 2u\,du, \qquad dy = 2v\,dv, \qquad dz = 2w\,dw.$$

Definition

$\alpha \in \Lambda^k(S^n)$ is even if it is invariant under each coordinate reflection. Let $\Lambda^k_e(S^n)$ denote the space of such forms.

Example

$$u^2 + v^4 w^2 \qquad u\,du \qquad uvw^2\,du \wedge dv$$

Theorem (YBK)

The change of coordinates induces a bijection between $\mathcal{P}_r\Lambda^k(T^n)$ and $\mathcal{P}_{2r+k}\Lambda^k_e(S^n)$.

Even and odd forms on the sphere

Definition

$\alpha \in \Lambda^k(S^n)$ is odd if it changes sign under each coordinate reflection. Let $\Lambda^k_o(S^n)$ denote the space of such forms.

Definition

Let u_N denote the product of the coordinate functions.

- In dimension 2, $u_N = uvw$.

Proposition

- *If α is even, then $*_{S^n}\alpha$ is odd, and vice versa.*
- *If α is even, then $u_N\alpha$ is odd, and vice versa.*

Proof.

- Reflections reverse orientation, which changes the sign of $*_{S^n}$.
- u_N is odd. □

Correspondences between forms on T^n and forms on S^n

Theorem (YBK)

Let $a \in \mathcal{P}_r\Lambda^k(T^n)$ and $\alpha \in \mathcal{P}_{2r+k}\Lambda^k_e(S^n)$ correspond to each other via the change of coordinates. Then for $r \geq 1$,

$$
\begin{aligned}
a \in \mathcal{P}_r^-\Lambda^k(T^n) \quad &\Leftrightarrow \quad \alpha \in *_{S^n}\mathcal{P}_{2r+k-1}\Lambda_o^{n-k}(S^n) \\
a \in \mathring{\mathcal{P}}_r\Lambda^k(T^n) \quad &\Leftrightarrow \quad \alpha \in u_N\mathcal{P}_{2r+k-n-1}\Lambda_o^{k}(S^n) \\
a \in \mathring{\mathcal{P}}_r^-\Lambda^k(T^n) \quad &\Leftrightarrow \quad \alpha \in u_N *_{S^n}\mathcal{P}_{2r+k-n-2}\Lambda_e^{n-k}(S^n).
\end{aligned}
$$

Explicit pointwise duality for $\mathcal{P}_r\Lambda^k(T^n)$ and $\mathring{\mathcal{P}}^-_{r+k+1}\Lambda^{n-k}(T^n)$

1. $a \in \mathcal{P}_r\Lambda^k(T^n)$,
2. $\alpha \in \mathcal{P}_{2r+k}\Lambda^k_e(S^n)$,
3. $*_{S^n}\alpha \in *_{S^n}\mathcal{P}_{2r+k}\Lambda^k_e(S^n)$,
4. $\beta = u_N *_{S^n}\alpha \in u_N *_{S^n}\mathcal{P}_{2r+k}\Lambda^k_e(S^n)$,
5. $b \in \mathring{\mathcal{P}}^-_{r+k+1}\Lambda^{n-k}(T^n)$.

Correspondences between forms on T^n and forms on S^n

Theorem (YBK)

Let $a \in \mathcal{P}_r\Lambda^k(T^n)$ and $\alpha \in \mathcal{P}_{2r+k}\Lambda^k_e(S^n)$ correspond to each other via the change of coordinates. Then for $r \geq 1$,

$$\begin{aligned}
a \in \mathcal{P}_r^-\Lambda^k(T^n) \quad &\Leftrightarrow \quad \alpha \in *_{S^n}\mathcal{P}_{2r+k-1}\Lambda^{n-k}_o(S^n)\\
a \in \mathring{\mathcal{P}}_r\Lambda^k(T^n) \quad &\Leftrightarrow \quad \alpha \in u_N\mathcal{P}_{2r+k-n-1}\Lambda^k_o(S^n)\\
a \in \mathring{\mathcal{P}}_r^-\Lambda^k(T^n) \quad &\Leftrightarrow \quad \alpha \in u_N *_{S^n}\mathcal{P}_{2r+k-n-2}\Lambda^{n-k}_e(S^n).
\end{aligned}$$

Explicit pointwise duality for $\mathcal{P}_r^-\Lambda^k(T^n)$ and $\mathring{\mathcal{P}}_{r+k}\Lambda^{n-k}(T^n)$

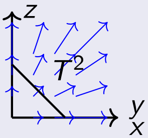
1. $a \in \mathcal{P}_r^-\Lambda^k(T^n)$,
2. $\alpha \in *_{S^n}\mathcal{P}_{2r+k-1}\Lambda^{n-k}_o(S^n)$,
3. $*_{S^n}\alpha \in \mathcal{P}_{2r+k-1}\Lambda^{n-k}_o(S^n)$,
4. $\beta = u_N *_{S^n}\alpha \in u_N\mathcal{P}_{2r+k-1}\Lambda^{n-k}_o(S^n)$,
5. $b \in \mathring{\mathcal{P}}_{r+k}\Lambda^{n-k}(T^n)$.

(optional slide) Alternate characterizations of $\mathcal{P}_r^-\Lambda^k(T^n)$

Definition

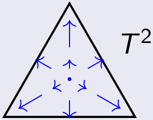
- Let X be the radial vector field

$$X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}.$$



- Let X_T be the projection of the radial vector field to T^n.

$$X_T = \left(x - \tfrac{1}{3}\right)\frac{\partial}{\partial x} + \left(y - \tfrac{1}{3}\right)\frac{\partial}{\partial y} + \left(z - \tfrac{1}{3}\right)\frac{\partial}{\partial z}.$$



- Let $i_X\colon \Lambda^k(T^2) \to \Lambda^{k-1}(T^2)$ denote contraction.

Definition (Arnold, Falk, and Winther)

$$\mathcal{P}_r^-\Lambda^k(T^n) := \mathcal{P}_{r-1}\Lambda^k(T^n) + i_{X_T}\mathcal{P}_{r-1}\Lambda^{k+1}(T^n).$$

(optional slide) Alternate characterizations of $\mathcal{P}_r^-\Lambda^k(T^n)$

$\mathcal{P}_r\Lambda^k(T^n)$ is the restriction of $\mathcal{P}_r\Lambda^k(\mathbb{R}^{n+1})$ to T^n. Likewise...

Definition (YBK)

$$\mathcal{P}_r^-\Lambda^k(\mathbb{R}^{n+1}) := i_X\mathcal{P}_{r-1}\Lambda^{k+1}(\mathbb{R}^{n+1})$$

Let $\mathcal{P}_r^-\Lambda^k(T^n)$ be the restriction of $\mathcal{P}_r^-\Lambda^k(\mathbb{R}^{n+1})$ to T^n.

Theorem (YBK)

The two definitions are equivalent.

Thank you